

M-3

Q:(1) What is the auxiliary equation?

Solution:

The equation $D^n y + P_1 D^{n-1} y + \dots + P_n y = 0$ — (1)
where P_1, P_2, \dots, P_n are constant is called linear
nth order homogeneous differential equation
with constant coefficients. If $y = e^{mx}$ is the solⁿ of
eqn (1), then $m^n e^{mx} + P_1 m^{n-1} e^{mx} + \dots + P_n e^{mx} = 0$
Cancelling the non-vanishing factor e^{mx} we get
 $m^n + P_1 m^{n-1} + \dots + P_n = 0$ which is called the
auxiliary equation of the differential eqn (1).

Q:(2) What is Linearly Independent?

Solution: Two functions $y_1(x)$ and $y_2(x)$ are
said to be LI if the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \text{ is not equal to}$$

zero.

Q:(3) Find the solution of $y'' - 3y' + 2y = 0$

Solution:

$$\text{Given that } y'' - 3y' + 2y = 0$$

This is a homogeneous, linear differential equation

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2 \text{ (Real and distinct roots)}$$

The general solution is

$$y(x) = C_1 e^x + C_2 e^{2x}$$

Q: (4) Define LD of two functions?

Solution:

Two functions $y_1(x)$ and $y_2(x)$ are said to be LD if the Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is equal to zero.

Q: (5) What is Euler-Cauchy Equations?

Solution:

The Euler Cauchy equations are ODEs of the form $ax^2y'' + axy' + by = 0$ ——— ①

with given constants a and b and unknown $y(x)$. We substitute $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

putting in the eqn ① we get

$$x^2 (m(m-1)x^{m-2}) + ax(m x^{m-1}) + bx^m = 0$$

$$\Rightarrow m(m-1)x^m + amx^m + bx^m = 0$$

$$\Rightarrow (m(m-1) + am + b)x^m = 0$$

Q: (6) Solve $y'' + y = 0$

Solution:

The Auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \sqrt{-1} = \pm i$$

Thus general solution is $y(x) = c_1 \cos x + c_2 \sin x$.

Q: (7) What is Kirchhoff's voltage law (KVL)?

Solution:

The algebraic sum of all the instantaneous voltage drops around any closed loop is zero.

voltage drop around across resistor $E_R = RI$

voltage drop around across inductor $E_L = L \frac{dI}{dt}$

voltage drop around across capacitor $E_C = \frac{Q}{C}$

Q: (8) Solve $y'' + 6y' + 9 = 0$

Solution:

The auxiliary equation is $m^2 + 6m + 9 = 0$

$$m^2 + 3m + 3m + 9 = 0$$

$$\Rightarrow m(m+3) + 3(m+3) = 0$$

$$\Rightarrow (m+3)(m+3) = 0$$

$$\Rightarrow m = -3, -3 \text{ (Real double roots)}$$

Hence general solution is

$$y(x) = (c_1 + c_2 x) \cdot e^{-3x}$$

Q: (9) Model the R-C circuit. Find the current due to constant E.

Solution: $E = E_0 = \text{constant}$

The equation is $RI + \frac{Q}{C} = E_0 = \text{constant}$

$$RI + \frac{I dt}{C} = E_0 \quad \left(\because I = \frac{dq}{dt}, q = \int I dt \right)$$

$$R \frac{dI}{dt} + \frac{I}{C} = 0, \quad \frac{dI}{dt} = -\frac{I}{RC}$$

$$\Rightarrow \frac{dI}{I} = -\frac{I}{RC} dt$$

Integrating on both side

$$\Rightarrow \int \frac{dI}{I} = -\frac{I}{RC} \int dt$$

$$\Rightarrow \log I = -\frac{I}{RC} \cdot t + C$$

$$\Rightarrow I = e^{-\frac{I}{RC} \cdot t + C}$$

Q. (10) solve $(4D^2 + 4\pi D + \pi^2)y = 0$

Solution:

The auxiliary equation is

$$4m^2 + 4\pi m + \pi^2 = 0$$

$$(2m + \pi)^2 = 0$$

$$\Rightarrow (2m + \pi)(2m + \pi) = 0$$

$$\Rightarrow m = -\pi/2, -\pi/2$$

Thus General solution is

$$y(x) = (C_1 + C_2 x) e^{-\pi/2}$$

- Long Types -

Q: (1) Solve $(D^2+4)y = \sin 3x + e^x + x^2$

Solution:

The given equation is $(D^2+4)y = \sin 3x + e^x + x^2$.

The auxiliary equation is $m^2+4=0$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m = \pm\sqrt{-4}$$

$$\Rightarrow m = \pm 2i \text{ (Complex roots)}$$

Hence the complementary function is

$$y_c = C_1 \cos 2x + C_2 \sin 2x \quad \text{--- (1)}$$

Now the particular Integral is

$$y_p = \frac{\sin 3x + e^x + x^2}{D^2+4}$$

$$= \frac{\sin 3x}{D^2+4} + \frac{e^x}{D^2+4} + \frac{x^2}{D^2+4}$$

$$= \frac{\sin 3x}{-3^2+4} + \frac{e^x}{(1)^2+4} + \frac{x^2}{4\left(1+\frac{D^2}{4}\right)}$$

$$= \frac{\sin 3x}{-9+4} + \frac{e^x}{1+4} + \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} \cdot x^2$$

$$= \frac{\sin 3x}{-5} + \frac{e^x}{5} + \frac{1}{4}\left(1 - \frac{D^2}{4} + \frac{D^4}{2 \cdot 16} - \dots\right)x^2$$

$$= \frac{1}{5}(e^x - \sin 3x) + \frac{1}{4}\left(x^2 - \frac{D^2}{4} \cdot x^2 + \dots\right)$$

$$= \frac{1}{5}(e^x - \sin 3x) + \frac{1}{4}\left(x^2 - \frac{1}{4}x^2\right)$$

$$= \frac{1}{5}(e^x - \sin 3x) + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)$$

Q: (2) Solve $(D^2 - 2D - 3)y = 2e^x - 10\sin x$ by undetermined coefficient (UC)

Solution:

The given differential equation is $(D^2 - 2D - 3)y = 2e^x - 10\sin x$

The auxiliary equation is $m^2 - 2m - 3 = 0$

$$\Rightarrow m^2 - 3m + m - 3 = 0$$

$$\Rightarrow m(m-3) + 1(m-3) = 0$$

$$\Rightarrow (m+1)(m-3) = 0$$

$$\Rightarrow m = -1, 3$$

The complementary function is

$$y_c = C_1 e^{-x} + C_2 e^{3x}$$

The UC functions are given by e^x and $\sin x$

$$\text{Let } y_p = A e^x + B \sin x + C \cos x$$

where A, B, C are undetermined coefficients

$$\Rightarrow y_p' = A e^x + B \cos x - C \sin x$$

$$y_p'' = A e^x - B \sin x - C \cos x$$

Now putting the values of y_p, y_p' and y_p'' in the given eqn we get

$$(A e^x - B \sin x - C \cos x) - 2(A e^x + B \cos x - C \sin x)$$

$$- 3(A e^x + B \sin x + C \cos x) = 2e^x - 10\sin x$$

$$\Rightarrow -4Ae^x + \sin x (-B + 2C - 3B) + \cos x (-C - 2B - 3C) = 2e^x - 10\sin x$$

$$\Rightarrow -4Ae^x + \sin x (2C - 4B) + \cos x (-2B - 4C) = 2e^x - 10\sin x$$

Now equating co-efficient of e^x , $\sin x$ and $\cos x$ from both the sides we get

$$-4A = 2, \quad 2C - 4B = -10, \quad -2B - 4C = 0$$

$$\Rightarrow A = \frac{2}{4} = -\frac{1}{2}, \quad C - 2B = -5, \quad B + 2C = 0$$

(Dividing 2) (Common -2)

Solving these we get $A = -\frac{1}{2}$, $B = 2$, $C = -1$

$$\therefore y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$$

$$\therefore y(x) = y_c + y_p$$

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{1}{2}e^x + 2\sin x - \cos x \text{ (Ans.)}$$

Q: (3) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Solution:

The Auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \sqrt{-1} = \pm i$$

The complementary function is

$$= A \cos x + B \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

Here particular integral = $u y_1 + v y_2$

$$u = \int \frac{-y_2 \cdot x}{y_1 \cdot y_2' - y_1' \cdot y_2} dx$$

$$u = \int \frac{-\sin x \cdot \operatorname{cosec} x \, dx}{\cos x (\cos x) - (-\sin x) \cdot \sin x}$$

$$= \int \frac{-\sin x \cdot \frac{1}{\sin x} \, dx}{\cos^2 x + \sin^2 x}$$

$$= - \int \frac{dx}{1} = -x$$

$$v = \int \frac{y_1 \cdot x}{y_1 \cdot y_2' - y_1' \cdot y_2} \, dx$$

$$= \int \frac{\cos x \cdot \operatorname{cosec} x \, dx}{\cos x (\cos x) - (-\sin x) \cdot \sin x}$$

$$= \int \frac{\cos x \cdot \frac{1}{\sin x} \, dx}{\cos^2 x + \sin^2 x}$$

$$= \int \cot x \, dx = \log \sin x$$

$$P.I. = \cos x(-x) + \sin x(\log \sin x)$$

Thus general solution is $y = C.F. + P.I.$

$$y(x) = A \cos x + B \sin x - x \cos x + \sin x (\log \sin x)$$

Q. (4) solve $(D^2 + 2D + 1)y = e^{-x} \log x$ by variation parameter.

Solution:

The given differential equation is $(D^2 + 2D + 1)y = e^{-x} \log x$

The auxiliary equation is $m^2 + 2m + 1 = 0$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1 \text{ (Repeated roots)}$$

$$y_c = (C_1 + C_2 x) e^{-x}$$

Here $y_1(x) = e^{-x}$, $y_2(x) = x \cdot e^{-x}$, $R(x) = e^{-x} \log x$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}$$

$$= e^{-x}(e^{-x} - x e^{-x}) + e^{-x} \cdot x e^{-x}$$

$$= e^{-2x} - x \cdot e^{-2x} + x \cdot e^{-2x} = e^{-2x}$$

Particular integral (P.I.) = $u \cdot y_1 + v \cdot y_2$

$$u = - \int \frac{R(x) \cdot y_2(x)}{W} dx$$

$$= - \int \frac{e^{-x} \log x \cdot x e^{-x}}{e^{-2x}} dx$$

$$= - \int x \cdot \log x dx$$

$$= - \left[\log x \int x dx - \left[\int \frac{d}{dx} (\log x) \cdot \int x dx \right] dx \right]$$

$$= - \left\{ \frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right\}$$

$$= - \frac{x^2}{2} \log x + \int \frac{x^2}{2} dx$$

$$= - \frac{x^2}{2} \log x + \frac{x^2}{4}$$

$$V = \int \frac{R(x) \cdot y_1(x)}{W} dx$$

$$= \int \frac{e^{-x} \cdot \log x \cdot e^{-x}}{e^{-2x}} dx$$

$$= \int \log x dx$$

$$= x \log x - x$$

$$y_p = u \cdot y_1 + v \cdot y_2$$

$$= \left(-\frac{x^2}{2} \log x + \frac{x^2}{4}\right) \cdot e^{-x} + (x \log x - x) \cdot x e^{-x}$$

$$\therefore y = (C_1 + C_2 x) e^{-x} + \left(-\frac{x^2}{2} \log x + \frac{x^2}{4}\right) \cdot e^{-x} + (x \log x - x) \cdot x e^{-x}$$

$$\Rightarrow y = (C_1 + C_2 x) e^{-x} + x^2 e^{-x} \left[-\frac{\log x}{2} + \frac{1}{4} + \log x - 1\right]$$

$$= (C_1 + C_2 x) e^{-x} + \left(\frac{\log x}{2} - \frac{3}{4}\right) \cdot x^2 e^{-x}$$

$$\Rightarrow y = e^{-x} \left\{ (C_1 + C_2 x) + \left(\frac{\log x}{2} - \frac{3}{4}\right) x^2 \right\}$$

Q. (5) Solve the differential equation

$$\frac{d^2 y}{dx^2} + a^2 y = \frac{a^2 R}{P} (l - x)$$

where a, R, P and l are constants subject to the conditions

$$y = 0, \frac{dy}{dx} = 0 \text{ at } x = 0$$

Solution:

The given differential equation is

$$\frac{d^2 y}{dx^2} + a^2 y = \frac{a^2}{P} R(l-x)$$

or $(D^2 + a^2)y = \frac{a^2}{P} R(l-x)$

The auxiliary equation is

$$m^2 + a^2 = 0$$

$$\Rightarrow m^2 = -a^2$$

$$\Rightarrow m = \sqrt{-a^2} = \sqrt{a^2} \cdot \sqrt{-1} = \pm ia$$

complementary function (C.F.) = $C_1 \cos ax + C_2 \sin ax$

$$P.I. = \frac{1}{D^2 + a^2} \cdot \frac{a^2}{P} R(l-x)$$

$$= \frac{a^2 R}{P} \cdot \frac{1}{a^2} \left[\frac{1}{1 + \frac{D^2}{a^2}} \right] (l-x)$$

$$= \frac{R}{P} \left[1 + \frac{D^2}{a^2} \right]^{-1} (l-x)$$

$$= \frac{R}{P} \left[1 - \frac{D^2}{a^2} \right] (l-x)$$

$$= \frac{R}{P} (l-x)$$

$$y(x) = C_1 \cos ax + C_2 \sin ax + \frac{R}{P} (l-x) \quad \text{--- (1)}$$

on putting $y=0$ and $x=0$ in eqn (1)

$$C_1 + \frac{R}{P} l = 0$$

$$\Rightarrow C_1 = -\frac{R}{P} l$$

on differentiating eqn (1) we get

$$\frac{dy}{dx} = -a C_1 \sin ax + a C_2 \cos ax - \frac{R}{P} \quad \text{--- (2)}$$

on putting $\frac{dy}{dx} = 0$ and $x=0$ we have

$$ae_2 = \frac{R}{p} = 0 \text{ or } e_2 = \frac{R}{a \cdot p}$$

on putting the values of G and e_2 in eqⁿ ①,

$$y = \frac{-R}{p} l \cos ax + \frac{R}{a \cdot p} \sin ax + \frac{R}{p} (l-x)$$

$$y = \frac{-R}{p} l \cos ax + \frac{R}{a \cdot p} \sin a \cdot x + \frac{R}{p}$$

$$y = \frac{R}{p} \left[\frac{\sin ax}{a} - l \cos ax + l - x \right]$$

① $\rightarrow (x-1) \left[\frac{R}{p} + x \cos ax + \frac{R}{p} \sin ax + \frac{R}{p} (l-x) \right]$

③ $\rightarrow \frac{R}{p} - \frac{R}{p}$