MODULE:- 1 AM-I

SHORT TYPES:-

Q:-(1) What is an asymptotes with a example?

Ans:- A straight line is said to be an asymptote of an infinite branch of a curve if as the point P recedes to infinity along the branch, the perpendicular distance of P from the straight line tends to zero.

EX:- The line $x=0$ and $y=0$ is an asymptotes of the curve $xy=1$.

We can write either $y=\frac{1}{x}$ or $x=\frac{1}{y}$

when $x \to 0$, $y \to \infty$

or $y \to 0$, $x \to \infty$.

(Q):-(2) Find the asymptotes $x^3+y^3-3axy=0$

Ans:- Divide term x^3 i.e. the highest power,

we get $1+\left(\frac{y}{x}\right)^3 - 3a\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=0$

Taking $x \to \infty$, we get $1+m^3=0$

$\Rightarrow (1+m)(1-m+m^2)=0$

$\Rightarrow m=-1$

other roots are Imaginary, next we have to calculate

$c = \lim_{x \to \infty}(y-mx) = \lim_{x \to \infty}(y+x)$

So, we write $x^3+y^3-3axy=0$

$$x+y=\frac{3axy}{x^2-xy+y^2}$$

$$\lim_{x\to\infty}(x+y)=\lim_{x\to\infty}\frac{3axy}{x^2-xy+y^2}$$

$$c=\lim_{x\to\infty}\left\{\frac{3a\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2}\right\}=\frac{3a(-1)}{1-(-1)+(-1)^2}=-a$$

(Q):-(3) Find the asymptotes $x^3+y^3=3ax^2$

<u>Solution:-</u>

Put $x=1$, $y=m$, we get

$\phi_3(m)=1+m^3$, $\phi_3'(m)=3m^2$, $\phi_2(m)=-3a$

Put $\phi_3(m)=0$

$1+m^3=0$, $(1+m)(1+m+m^2)=0$

$m+1=0$; $m=-1$

find 'c' using relation

$c\,\phi_3'(m)+\phi_2(m)=0$

$c(3m^2)-3a=0$

for $m=-1$, $c=a$,

Hence $y+x=a$ is the asymptotes.

(Q) :- (4) Find the radius of curvature of the curve $y = 3 + 2x - x^2$

Solution :-

$$y = 3 + 2x - x^2$$

$$\frac{dy}{dx} = 2 - 2x, \quad \frac{d^2y}{dx^2} = -2$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{\left[1 + (2-2x)^2\right]^{3/2}}{|-2|}$$

$$= \frac{\left[1 + (2-2x)^2\right]^{3/2}}{2}$$

(Q) :- (5) Find the radius of curvature at any point of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$

Solution :-

$$x = a(t + \sin t); \quad y = a(1 - \cos t)$$

$$x' = a(1 + \cos t); \quad y' = a \sin t$$

$x'' = -a\sin t$; $y'' = a\cos t$.

$$\rho = \frac{[x'^2 + y'^2]^{3/2}}{[x'y'' - y'x'']}$$

$$= \frac{[a^2(1+\cos t)^2 + a^2\sin^2 t]^{3/2}}{a^2(\cos t + 1)}$$

$$= 4a\cos\frac{1}{2}t.$$

(Q) (6) Find the radius of curvature at the origin

$$x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0$$

Solution:

Here x-axes is the tangent to the origin.

Divide y on both the sides

$$x^2 \cdot \frac{x^2}{y} - y^3 + x \cdot \frac{x^2}{y} - y^2 + \frac{x^2}{y} - y + 1 = 0$$

Taking the limit $x \to 0$ and $y \to 0$

$$2\rho + 1 = 0$$

$$\Rightarrow 2\rho = -1$$

$$\Rightarrow \rho = -\frac{1}{2}.$$

(Q):(7) verify that v satisfies laplace's equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

if $v = x^2 + y^2 - 2z^2$

Solution:-

$$v = x^2 + y^2 - 2z^2$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial^2 v}{\partial x^2} = 2$$

$$\frac{\partial v}{\partial y} = 2y; \quad \frac{\partial^2 v}{\partial y^2} = 2$$

$$\frac{\partial v}{\partial z} = -4z; \quad \frac{\partial^2 v}{\partial z^2} = -4$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 2+2-4=0$$

(Q):(8) Evaluate $\int_0^\infty e^{-x^2} dx$

Solution:-

$$I = \int_0^\infty e^{-x^2} dx$$

let $x^2 = t \Rightarrow x = t^{1/2}$

$$dx = \frac{1}{2} t^{-1/2} dt$$

when $x=0$; $t=0$

When $x=\infty$; $t=\infty$

So $I = \int_0^\infty e^{-t} \frac{1}{2} t^{-1/2}\, dt$

$= \frac{1}{2}\int_0^\infty e^{-t} . t^{-1/2}\, dt$

$= \frac{1}{2}\int_0^\infty e^{-t} . t^{\frac{1}{2}-1}\, dt$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{2}$

Q:- (9) Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x\, dx$

Solution:- $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$

where $m=4$; $n=3$

$= \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\cdot 1!}{2\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)} = \frac{2}{35}.$$

Q:(10) Write down the method for finding extrema of $f(x,y)$.

Solution:-

1. Solving $f_x=0$ and $f_y=0$ yields critical or stationary point P of f.

2. Calculate $r=f_{xx}$, $s=f_{xy}$, $t=f_{yy}$ at the critical point P.

3. (a) maximum: If $rt-s^2>0$ and $r>0$ then f has a minimum at P.

(b) minimum: If $rt-s^2>0$ and $r>0$ then f has a maximum at P.

(c) Saddle point: If $rt-s^2<0$ then f has neither maximum nor minimum.

(d) failure case: If $rt-s^2=0$, further investigation needed.

LONG TYPES

Q:-(1) find the asymptotes of
$x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$

Solution:
put $x=1$, and $y=m$ we get

$\phi_3(m) = 1-m-m^2+m^3$

$\phi_3'(m) = -1-2m+3m^2$

$\phi_3''(m) = -2+6m$

$\phi_2(m) = 2-4m^2+2m$

$\phi_2'(m) = -8m+2$

$\phi_1(m) = 1+m$

$\phi_0(m) = 1$

put $\phi_3(m)=0$, $1-m-m^2+m^3=0$

$m=-1, 1, 1$

for $m=-1$, for finding 'c' using relation

$c\phi_3'(m) + \phi_2(m)=0$

$c(-1-2m+3m^2)+(2-4m^2+2m)=0$,

$$c = \frac{4m^2-2m-2}{3m^2-2m-1}$$

for $m=-1$, $c=1$, the required asymptote to the $y=-x+1$

for $m=1$, for finding 'c' using relation

$$\frac{c^2}{\lfloor 2}\phi_3''(m)+c\phi_2'(m)+\phi(m)=0$$

$$\frac{c^2}{2}(-2+6m)+c(-8m+2)+(1+m)=0$$

for $m=1$, $2c^2-6c+2=0$

$$c=\frac{3\pm\sqrt{5}}{2}$$

the required asymptote is $y=x+\frac{3\pm\sqrt{5}}{2}$

Hence the asymptotes are $y=-x+1$, $y=x+\frac{3+\sqrt{5}}{2}$ and $y=x+\frac{3-\sqrt{5}}{2}$

Q:-(2) Show that the radius of curvature of the curve $y^2=\frac{x^2(a+x)}{(a-x)}$ at the origin is $\pm a\sqrt{2}$.

Solution:-

$$y^2=\frac{x^2(a+x)}{(a-x)}$$

$$\Rightarrow y=\pm x\sqrt{\frac{a+x}{a-x}}$$

$$\Rightarrow y=\pm\sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}}$$

$$\Rightarrow y = \pm x \frac{\sqrt{(a+x)^2}}{\sqrt{a^2-x^2}}$$

$$\Rightarrow y = \pm x \frac{(a+x)}{\sqrt{a^2-x^2}}$$

$$\Rightarrow y = \pm x \frac{(a+x)}{(a^2-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow y = \pm x (a+x) \cdot (a^2-x^2)^{-1/2}$$

$$\Rightarrow y = \pm x \left(\frac{a+x}{a}\right) \cdot \left(1-\frac{x^2}{a^2}\right)^{-1/2}$$

$$\Rightarrow y = \pm x\left(1+\frac{x}{a}\right)\left(1+\frac{1}{2}\frac{x^2}{a^2}+\cdots\right)$$

$$\Rightarrow y = \pm\left(x+\frac{1}{a}x^2+\cdots\right)$$

When $y = x+\frac{1}{a}x^2$

This is of the form

$y = px + q\cdot\frac{x^2}{2}+\cdots$

Here $p=1$; $q=\frac{2}{a}$

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{(2/a)} = \sqrt{2}a$$

When $y = -x-\frac{1}{a}x^2$

Here $p=-1$, $q=\frac{-2}{a}$

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\left(\frac{-2}{a}\right)} = -\sqrt{2}a$$

Hence $f = \pm\sqrt{2}a$.

Q:-(3) If $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Solution:-

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2y$$

$$= -y(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$

$$\frac{\partial u}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2z$$

$$= -z(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

Adding we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Q:-(4) Use the Taylor's theorem to expand $f(x,y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$.

Solution:-

$$f(x,y)=x^2+xy+y^2$$

Differentiating w.r.t "x" and "y" we get $f_x=2x+y$, $f_y=x+2y$

$f_{xx}=2$, $f_{xy}=1$, $f_{yy}=2$

$f_{xxx}=0$, $f_{xxy}=0$, $f_{yyy}=0$

The Taylor's series expansion of $f(x,y)$ in powers of $(x-1)$ and $(y-2)$ is

$$f(x,y)=f(1,2)+\left[(x-1)f_x(1,2)+(y-2)f_y(1,2)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2)+2(x-1)(y-2)f_{xy}(1,2)+(y-2)^2 f_{yy}(1,2)\right]$$

$$+\frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,2)+3(x-1)^2(y-2)f_{xxy}(1,2)+3(x-1)(y-2)^2 f_{yyx}(1,2)+(y-2)^3 f_{yyy}(1,2)\right]$$

Here $f(1,2)=7$, $f_x(1,2)=4$, $f_y(1,2)=5$,
$f_{xy}(1,2)=1$, $f_{xx}=f_{yy}=2$ etc.

Substituting these values

$$f(x,y)=7+4(x-1)+5(y-2)$$

$$+\frac{1}{2!}\left[2(x-1)^2+2(x-1)(y-2)+2(y-2)\right.$$

Q:-(5) Find the maximum and minimum values of $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$

Solution:-

Here $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$

$f_x=3x^2+3y^2-30x+72$

$f_y=6xy-30y$

The stationary points are given by

$f_x=0$ and $f_y=0$

$f_y=6xy-30y=0$

$=6y(x-5)=0$

$y=0$ or $x=5$,

$f_x=3x^2-30x+72=0$

for $y=0$, $3x^2-30x+72=0$

$x=6$ or 4.

For $x = 5$, $75 + 3y^2 - 150 + 72 = 0$; $y = \pm 1$

Thus the stationary points are $(6,0)$ $(4,0)$ $(5,1)$ and $(5,-1)$

To determine the nature of these points

$A = f_{xx} = 6x - 30$

$B = f_{xy} = 6y$

$C = f_{yy} = 6x - 30$

So $AC - B^2 = (6x-30)(6x-30) - 36y^2$

$= 36((x-5)^2 - y^2)$

1. At the stationary point $(6,0)$ we have $AC - B^2 = 36 > 0$ and $A = 36 - 30 = 6 > 0$

so $(6,0)$ is a minimum point of the given function f and the minimum value of f at $(6,0)$ is $= 6^3 + 0 - 15(36) + 72(6) = 108$

2. At $(4,0)$: $A = 24 - 30 = -6 < 0$

$AC - B^2 = 36 > 0$

So a maximum occurs at the point $(4,0)$

The maximum value of "f" at $(4,0)$ is 112.

3. At $(5,1)$, $A = 0$, $AC - B^2 = -36 < 0$

So it is a saddle point

4. At $(5,-1)$, $A = 0$, $AC - B^2 = -36 < 0$

So $(5,-1)$ is a saddle point.