Mechanical Vibrations
[MV]

Module -IV

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OBJECTIVE-TYPE QUESTIONS

(1) Infinite number of degrees-of-freedom system means
  (a) maximum number of coordinates to specify their configuration.
  (b) infinitely large number of coordinates to specify their configuration.
  (c) minimum number of coordinates to specify their configuration
  (d) infinite number of natural frequencies of the system.

(2) In the vibration of continuous system for analysis of problems,
  (a) the knowledge of partial differential equation is very much essential.
  (b) constant boundary and initial conditions
  (c) the knowledge of partial differential equation is very much essential and constant boundary conditions and initial conditions
  (d) all of the above cases

(3) The one-dimensional wave for lateral vibration of a string is given by
  (a) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{\rho a^2} \frac{\partial^2 y}{\partial t^2}$
  (b) $\frac{\partial^2 y}{\partial x^2} = \rho a^2 \frac{\partial^2 y}{\partial t^2}$
  (c) $\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$
  (d) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2}$

(4) Examples of continuous systems are
  (a) spring-mass system
  (b) spring-mass-damper system
  (c) beams, rods, cables, plates
  (d) all of the above cases

(5) in case of continuous systems
  (a) finite number of coordinates specify their configuration
  (b) infinitely large number of coordinates specify their configuration
  (c) finite number of natural frequencies specify their configuration
  (d) none of the above cases

(6) longitudinal vibration of rods or bars is given by
  (a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$
  (b) $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$
  (c) $\frac{\partial^2 u}{\partial x^2} \frac{1}{a^2} = \frac{\partial^2 u}{\partial t^2}$
  (d) $\frac{G}{\rho} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$

(7) Torsional vibration of uniform shaft or rods is given by
  (a) $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 \theta}{\partial t^2} (x, t)$
  (b) $\frac{\partial^2 \theta}{\partial x^2} (x, t) = \frac{\partial^2 \theta}{\partial t^2}$
  (c) $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$
  (d) $\frac{\partial^2 \theta}{\partial t^2} (x, t) = \frac{1}{a^2} \frac{\partial^2 \theta}{\partial t^2} (x, t)$

(8) The general solution for transverse vibration of beams is given by
  (a) $y(x, t) = C \cos cx + D \sin cx$
  (b) $y(x, t) = A \cos h\, cx + B \sin h\, cx$
  (c) $A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$
  (d) $y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$

Answers

(1) b	(2) c	(3) d	(4) c	(5) b	(6) a
(7) d	(8) d				

SHORT-TYPE

Questions & Answers

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Q. What is a continuous system ?

How it is different from Multi-degree freedom system ?

Answer :

Rods, beams, and other structural components on the other hand are considered as continuous systems which have an infinite number of degrees of freedom. The vibration of such systems is governed by partial differential equations which involve variables that depend on time as well as the spatial coordinates.

Multi-degree-of-freedom (multi-DOF) systems are defined as those requiring two or more coordinates to describe their motion. This excludes continuous systems, which theoretically have an infinite number of freedoms.

Q. What are the different boundary and initial conditions for solving a differential equation in Continuous system ?

Ans:

1. Boundary and initial conditions In case of partial differential equations, the unknown value of constants can be determined by applying either geometric or natural or both boundary conditions.

2. Geometric boundary conditions These are due to geometric compatibility.

For example, if the bar is fixed at both the ends, the displacement and slope will be zero.

3. Natural boundary conditions These are due to force and moments.

For example, if the bar is hinged at one end, the bending moment at the hinged end will be zero and so on, whereas the initial conditions are related to time.

LONG -TYPE

Questions & Answers

Bijan Kumar Giri

1.a) Deduce the differential equation of lateral vibration of a string .

b) Also, find the solution of the above differential equation.

Answer :

Let us consider a string subjected to a transverse (lateral) vibration under tension 'T' of length 'L' as shown in Fig. 9.1(a) and let 'ρ' be the mass per unit length.

Assume that tension 'T' is large and is constant throughout its length 'L' also the amplitude of transverse vibration of the string is very small. For very small displacements of the string, $\sin\theta_1 = \tan\theta_1 = \theta_1$.

Let us consider a small element of length 'dx' at a distance 'x' from the y-axis as shown in Fig. 9.1(a). Let this element be displaced through a distance 'y' from the equilibrium position; then the FBD as shown in Fig. 9.1(b).

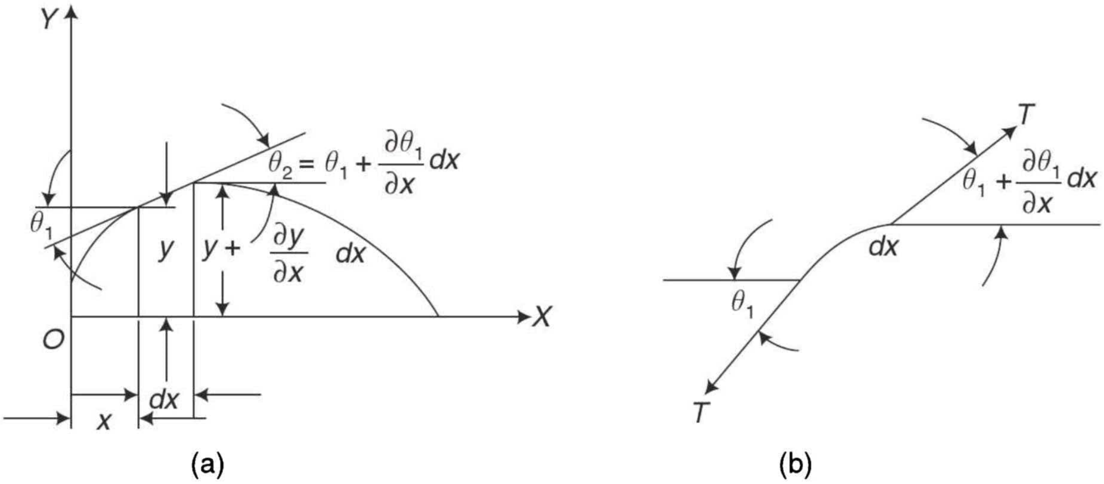

Fig. 9.1 *String in lateral vibration*

From FBD, let 'θ_1' be the angle subtended between the tangent to the small elemental string and normal to the elemental string at the left side of the string. Similarly, 'θ_2' be the angle subtended between the tangent to the elemental string and normal to the string, at the right side of the elemental string.

From geometry of the Fig. 9.1(b) in FBD, $\tan\theta_1 = \frac{\delta y}{\delta x}$.

If 'θ_1' is very small, $\tan\theta_1 = \theta_1$

$$\therefore \quad \theta_1 = \frac{\delta y}{\delta x}, \text{ and } \tan\theta_2 = \frac{\delta y}{\delta x} + \frac{\delta}{\delta x}\left(\frac{\delta y}{\delta x}\right)dx,\ \theta_2 = \theta_1 + \frac{\delta \theta_1}{\delta x}dx \qquad \text{...9.1}$$

since $\theta_1 = \frac{\delta y}{\delta x}$.

Resolving the forces along y-axis,

$$T\sin\theta_1 + \left(\frac{\partial \theta_1}{\partial x}dx\right) - T\sin\theta_1 = \text{Mass} \times \text{Acceleration}$$

$$T\left(\theta_1 + \frac{\partial \theta_1}{\partial x}dx\right) - T\theta_1 = \rho \cdot dx\,\frac{\partial^2 y}{\partial t^2},$$

where $\rho.dx$ = Mass of the small element of length 'dx'

$$T\left(\theta_1 + \frac{\partial \theta_1}{\partial x} dx\right) - T\theta_1 = \rho.dx \frac{\partial^2 y}{\partial t^2}, \ T\frac{\partial \theta_1}{\partial x} dx = \rho.dx \frac{\partial^2 y}{\partial t^2}, \ T\frac{\partial \theta_1}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}.$$

$$\frac{T}{\rho}\left(\frac{\partial \theta_1}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2}, \quad \frac{T}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2}, \left(\text{since } \theta_1 = \frac{\partial y}{\partial x}\right).$$

$$\frac{T}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2} \qquad \text{...9.2}$$

Let $\qquad a^2 = \dfrac{T}{\rho}$

The equation can be written as,

$$a^2\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2}, \ \frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{...9.3}$$

This is a one-dimensional wave equation for lateral vibration of string and the constant 'a' as the wave propagation velocity.

This equation has four arbitrary constants and can be solved by boundary and initial conditions.

Solution of wave equation The lateral deflection 'y' along the string is a function of the variables 'x' and 't'. So it can be written as $y = y(x, t)$. ...9.4

Let us assume the harmonic mode of vibration as the system is undamped.

Thus, solution of Eq. 9.3 can be written as $y(x, t) = X(x)\,T(t)$. ...9.5

Substituting the above solution in Eq. 9.3, we get

$$\frac{a^2}{X}\cdot\frac{d^2X}{dx^2} = \frac{1}{T}\cdot\frac{d^2T}{dt^2} \qquad \text{...9.6}$$

In this equation, LHS is a function of 'x' alone and RHS is a function of 't' alone. The above two can only be equal if each of the above equations is a constant. These constants will be 'zero', 'negative' or it may be 'positive'. If we consider 'zero' or 'positive constant' then there is no vibratory motion which is contrary to our observations for the practical systems. So we put it equal to some constant Z^2.

$$\frac{d^2X}{dx^2} + \left(\frac{Z}{a}\right)^2 X = 0 \text{ and } \frac{d^2T}{dt^2} + T^2 = 0 \qquad \text{...9.7}$$

The solutions of the above two equations are

$$X(x) = A\cos\left(\frac{Z}{a}\right)x + B\sin\left(\frac{Z}{a}\right)x, \ T(t) = C\cos Zt + D\sin Zt.$$

The general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{Z}{a}\right)x + B_n\sin\left(\frac{Z}{a}\right)x\right)[C_n\cos Zt + D_n\sin Zt] \qquad \text{...9.8}$$

In this equation, 'Z' is the frequency of vibration. The values of arbitrary parameters A_n, B_n, C_n and D_n in the above equation, can be determined by assuming boundary and initial conditions.

1. Boundary conditions Let us assume the string is fixed at both ends, i.e.
$y(0, t) = 0$ and $y(L, t) = 0$ ...9.9

2. Initial conditions Assuming the initial displacement and velocity as,

at $t = 0, \quad y(x, 0) = S(x)$

at $t = 0, \quad y(x, 0) = V(x)$ ...9.10.

Using these boundary conditions of equations 9.9 and 9.10 in Eq. 9.8, we have

$$y(0, t) = A_n(C_n \cos Zt + D_n \sin Zt) \text{ gives } A_n = 0$$

$$y(\mathrm{L}, t) = B_n \sin\left(\frac{Z}{a}\right) L(C_n \cos Zt + D_n \sin Zt), \quad \text{if } B_n \neq 0$$

which gives $\sin\left(\frac{Z}{a}\right) L = \sin n\,\pi = 0$ ...9.11.

This equation is called frequency equation.

$$\frac{Z_n}{a} L = n\pi,\ Z_n = \frac{n\pi a}{L} \left(\because a^2 = \frac{T}{\rho}\right), \text{ so frequency } Z_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \text{ rad/s} \qquad ...9.12$$

Normal mode shape can be written as $X(x) = \sin \frac{n\pi x}{L}$, $n = 1, 2, 3$ ...9.13

Each 'n' represents a mode of vibration example for $n = 1$ (first mode) $n = 2$ (second mode) and so on. Equation 9.8 can be written as

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L}\ C_n \cos Z_n t + D_n \sin Z_n t \qquad ...9.14$$

The values of constants 'C_n' and 'D_n' can be determined from initial conditions, i.e. displacement is $s(x)$ at $t = 0$ and velocity is $v(x)$ at $t = 0$.

Applying initial conditions for above equation, $s(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}$ ...9.15

$$v(\dot{x}) = \sum_{n=1}^{\infty} \ Z_n D_n \sin \frac{n\pi x}{L} \qquad ...9.16$$

Multiply the equations 9.15 and 9.16 each by $\sin \frac{m\pi x}{L}$, where $m = 1, 2, 3...$

and integrate from $x = 0$ to L.

Thus, $$\int_0^L s(x) \sin \frac{m\pi x}{L}\, dx = \int_0^L C_n \left(\sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\right) dx$$

$\sin \frac{n\pi x}{L}$ and $\sin \frac{m\pi x}{L}$ are orthogonal functions and the value of the above integral will be zero except when $m = n$.

Replacing $m = n$ for nonzero value of 'C_n', we get

$$\int_0^L s(x) \sin \frac{n\pi x}{L}\, dx = \int_0^L C_n \sin^2 \frac{n\pi x}{L}\, dx$$

$$\int_0^L s(x) \sin \frac{n\pi x}{L}\, dx = C_n \int_0^L \frac{1}{2}\left(1 - \cos^2 \frac{n\pi x}{L}\right) dx$$

So $$C_n = \frac{2}{L}\int_0^L s(x)\sin\frac{n\pi x}{L}\,dx \qquad \text{...9.17}$$

Similarly considering Eq. 9.16,

$$\int_0^L v(x)\sin\frac{m\pi x}{L}\,dx = Z_n D_n \int_0^L \sin\frac{n\pi x}{L}\cdot\sin\frac{m\pi x}{L}\,dx.$$

For nonzero value of D_n replace $m = n$

$$\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx = Z_n D_n \int_0^L \sin^2\frac{n\pi x}{L}$$

$$D_n = \frac{2}{Z_n L}\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx. \qquad \text{...9.18}$$

Q. Derive the differential equation of Longitudinal vibration of a rod or bar .

Ans : Let us consider a prismatic bar of length 'L' subjected to longitudinal vibration as shown in Fig. 9.2(a). Let 'A' be the cross-sectional area of the bar, 'E' be the Young's modulus of the materials, 'ρ' be the density of the material, and 'm' be the mass per unit length.

Let us assume that the bar should be thin and of uniform cross-section throughout of its length and subjected to axial force 'F' and there will be displacements 'u' along the rod that will be a function of both positions 'x' and time 't', because the rod has an infinite number of natural modes of vibration. The distribution of the displacements will differ with each mode as shown in Fig. 9.2(a).

Let us consider a small elemental length 'dx' at a distance 'x' from the left end and 'F' be the axial force on a small elemental length. The force on the other side, i.e. right side of small elemental length is equal to $\left(F + \frac{\partial F}{\partial x}dx\right)$.

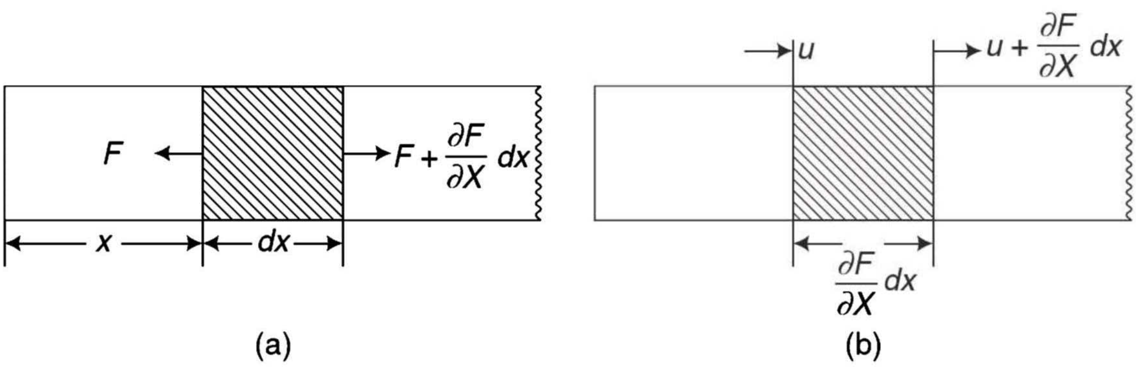

Fig. 9.2 *Longitudinal vibration of rods*

If 'u' is the displacement at a distance 'x' from the left side and $\left(u + \frac{\partial u}{\partial x}dx\right)$ displacement at a distance $x + dx$ at the right side of small elemental length. Now it is clear that from FBD as shown in Fig. 9.2(b), due to these axial forces on the small elemental length 'dx' there is a changed length by an amount equal to $\left(u + \frac{\partial u}{\partial x} dx - u\right)$ $= \left(\frac{\partial u}{\partial x} dx\right)$.

We know from mechanics of materials, when an element or a body is subjected either to the tension or compression, it undergoes stress, strain and deformation.

By definition of strain ($\in$) = Change in length /original length

$$\therefore \qquad \in = \frac{\frac{\partial u}{\partial x} dx}{dx} = \frac{\partial u}{\partial x} \qquad ...9.19$$

Net force acting on the small element,

$$\left(F + \frac{\partial F}{\partial x} \partial x\right) - F = (\text{Mass}) \times (\text{Acceleration of the element})$$

$$= dm \times \frac{\partial^2 u}{\partial t^2}, \quad \text{where } dm = \text{Mass of the small elemental length}$$

$$\frac{\partial F}{\partial x} dx = (\rho dx A)\left(\frac{\partial^2 u}{\partial t^2}\right) \qquad ...9.20$$

ρ = Density and $dx\ A$ = Volume of the small elemental length

We know that definition of stress (σ) equal to load /area or $\sigma = \frac{F}{A}$ or $F = \sigma A$.

$$\frac{\partial F}{\partial x} = \frac{\partial \sigma}{\partial x} A, \left(\frac{\partial F}{\partial x}\right) dx = \left(\frac{\partial \sigma}{\partial x}\right) dx\ A \qquad ...9.21$$

Equation 9.20 can be written with the help of above equation as

$$\left(\frac{\partial \sigma}{\partial x}\right) dxA = (\rho dxA)\left(\frac{\partial^2 u}{\partial t^2}\right) \qquad ...9.22$$

According to Hooke's law, stress $\propto$ strain 'within' elastic limit, i.e. $\sigma \propto \in$, $\sigma = E\in$ or $E = \frac{\sigma}{\in}$, $\frac{\text{Stress}}{\text{Strain}} = E$, where E = Young's modulus, $\sigma = \in E$, $\left(\frac{\partial \sigma}{\partial x}\right) dxA = \left(\frac{\partial^2 u}{\partial t^2}\right) dxAE$

...9.23

With the help of Eq. 9.22 and Eq. 9.23,

we get, $$\left(\frac{\partial \in}{\partial x}\right) dxAE = (\rho dxA)\left(\frac{\partial^2 u}{\partial t^2}\right)$$

But $\left(\in = \frac{\partial u}{dx}\right)$ [Eq. 9.19]

$$\left(\frac{E}{\rho}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial t^2}\right), \frac{E}{\rho}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 u}{\partial t^2}, \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \text{ where } a^2 = E/\rho$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...9.24}$$

This is the wave equation which is identical to Eq. 9.3 or $\left(\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}\right)$.

The general solution will be same as in the previous case of lateral vibrations.

A solution of the form is as in $u(x, t) = X(x)\, T(t)$

So $\qquad X(x) = A \sin \frac{Z_n x}{a} + B \cos \frac{Z_n x}{a}$, $T(t) = C \sin Z_n t + D \cos Z_n t$

will result into the general solution as

$$u(x, t) = \sum_{n=1}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right) (C \sin Z_n t + D \cos Z_n t) \qquad \text{...9.25}$$

EXAMPLE 9.1

Derive the frequency equation of longitudinal vibration for a free-free beam with zero initial displacement.

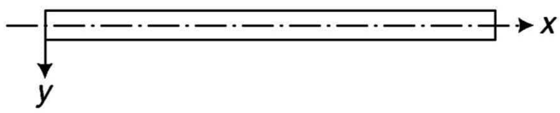

Fig. p-9.1 *Longitudinal vibration of a beam*

Solution The system is as shown in Fig. p-9.1.

We know that the general solution of longitudinal vibration of a uniform bar is given by Eq. 9.25

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right) (C \sin Z_n t + D \cos Z_n t).$$

where $a = \sqrt{\frac{E}{\rho}}$ and $Z_n = 2\pi f_n$; Z_n is the natural frequency.

The boundary conditions for the above particular system (free-free beam with zero initial displacement) are $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$ and $\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$ (for free end on both ends, strain is zero).

Differentiating the above equation (9.25) w.r.t. 'x' partially and applying these boundary conditions to the general solution, we get

$$\left(\frac{\partial u}{\partial x}\right) = \left(A \frac{Z_n}{a} \cos \frac{Z_n}{a} x - B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right) (C \sin Z_n t + D \cos Z_n t) \qquad \text{...9.26}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = A \frac{Z_n}{a} (C \sin Z_n t + D \cos Z_n t) \quad \therefore A = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \left(-B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right) (C \sin Z_n t + D \cos Z_n t),$$

$$\left(-B \frac{Z_n}{a} \sin \frac{Z_n}{a} L\right) (C \sin Z_n t + D \cos Z_n t)$$

By using solution of wave equations 9.17 and 9.18, we can determine the values of constants 'C' and 'D' from initial conditions.

So $\sin \frac{Z_n}{a} L = 0$, $\sin n\pi$, $Z_n = \frac{n\pi a}{L}$, $n = 1, 2, 3...$

We know that $Z_n = 2\pi f_n$, $2\pi f_n = \frac{n\pi a}{L}$

Therefore, the natural frequency $f_n = \frac{n}{2L} a$, but $a = \sqrt{\frac{E}{\rho}}$

$\therefore \quad f_n = \frac{n}{2L}\sqrt{\frac{E}{\rho}}$, '$n$' represent the order of the mode.

EXAMPLE 9.2

Derive an expression for the free longitudinal vibration of a uniform bar of length 'L', one end of which is fixed and the other end is free as shown in Fig. p-9.2.

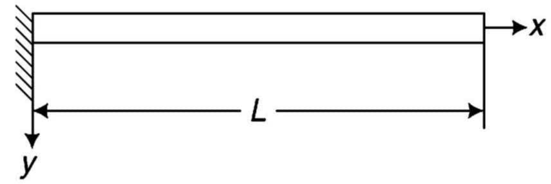

Fig. p-9.2 *Uniform bar*

Solution We know that the general solution of longitudinal vibration of a uniform bar is given by Eq. 9.25.

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

The boundary conditions for above particular system (one end of which is fixed and other end is free) are

$(u)_{x=0} = 0$, (displacement is zero at fixed end) and

$\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$, (strain is zero at free end).

Differentiating Eq. 9.26 w.r.t. 'x' partially, we get

$$\left(\frac{\partial u}{\partial x}\right) = \left(A \frac{Z_n}{a} \cos \frac{Z_n}{a} x - B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t) \quad ...9.27$$

Applying the boundary conditions to the general solution of Eq. 9.1, we have $B = 0$

$$0 = A \frac{Z_n}{a} \cos \frac{Z_n}{a} L(C \sin Z_n t + D \cos Z_n t) \text{ or } \cos \frac{Z_n}{a} L = 0 = \cos \frac{n\pi}{2},$$

where $n = 1, 3, 5...$

And $\quad A \neq 0.\ \frac{Z_n}{a} L = \frac{n\pi}{2}$, $\quad Z_n = \frac{n\pi a}{2L}$ $\quad$ But $Z_n = 2\pi f_n$, $2\pi f_n = \frac{na\pi}{2L}$

$\therefore \quad f_n = \frac{n}{4L}\sqrt{\frac{E}{\rho}} \quad \therefore a = \sqrt{\frac{E}{\rho}}$

The general solution of longitudinal vibration of a uniform bar can be written as

$$u(x, t) = \sum_{n=1,3,5.}^{\infty} \sin \frac{nx\pi}{2L}\left(C \sin \frac{na\pi}{2L} t + D \cos \frac{na\pi}{2L} t\right).$$

EXAMPLE 9.3

A bar of uniform cross-section having length '*L*' is fixed at both ends as shown in Fig. p-9.3. A bar is subjected to longitudinal vibrations having a constant velocity 'v_0' at all points. Derive suitable mathematical expression of longitudinal vibrations in the bar.

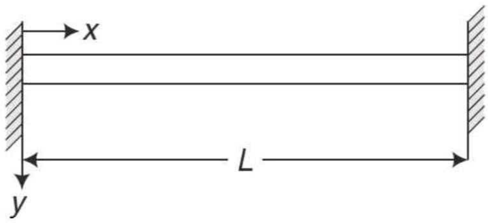

Fig. p-9.3 *Uniform bar*

Solution As we know that the general solution of longitudinal vibration of a uniform bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \operatorname{Cos} \frac{Z_n}{a} x\right) (C \sin Z_n t + D \cos Z_n t)$$

The boundary conditions, for the above particular system (fixed at both ends) are $x = 0$, displacement = 0,

i.e. $u(0, t) = 0, \quad u(L, t) = 0$

By using the first boundary condition, in the above general solution of longitudinal vibration of a uniform bar (9.25), we get

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \sin \frac{n\pi x}{L} (C \sin Z_n t + D \cos Z_n t)$$

$$B = 0$$

And by using second boundary conditions, we have $\frac{Z_n L}{a} = 0 = \sin n\pi$

$$n = 1,2,3\ldots, \text{ but } Z_n = \frac{n\pi a}{L}, \; a = \sqrt{\frac{E}{\rho}}$$

Substituting these value of Z_n in Eq. 9.20, we get

Again the initial conditions are $u(x, 0) = 0, \quad \dot{u}(x, 0) = V_0$

By using the first initial condition in the above general solution, we get

$$0 = \sum_{n=1,2,3,\ldots}^{\infty} \sin \frac{n\pi x}{L} \cdot D, \; D = 0$$

Then the equation is $u(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} \sin \frac{n\pi x}{L} \cdot C \sin Z_n t$

By using the second initial condition in the above general solution, we get

Then the equation is $\dot{u}(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} \sin \frac{n\pi x}{L} \cdot C Z_n \cos Z_n t$

$$\dot{u}(x, 0) = \sum_{n=1,2,3,\ldots}^{\infty} C Z_n \sin \frac{n\pi x}{L} = V_0$$

or $$C = \frac{2}{n\pi a} \int_0^L V_0 \sin \frac{n\pi x}{L} dx \text{ (Eq. 9.18)} \quad C = \frac{2 V_0 L}{n^2 \pi^2 a}(1 - \cos n\pi)$$

So $$C = \frac{4 V_0 L}{n^2 \pi^2 a} \text{ when } n = 1, 3, 5\ldots \quad \text{and } C = 0 \text{ when } n = 2, 4, 6\ldots$$

Finally, the required expression can be written as

$$u(x, t) = \frac{4V_0L}{\pi^2 a} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} \sin\frac{n\pi x}{L} \sin\frac{n\pi a}{L} t$$

EXAMPLE 9.4

Determine the normal function for free longitudinal vibration of a uniform bar of length '*L*' and uniform cross-section. Both ends of the bar are fixed as shown in Fig. p-9.4.

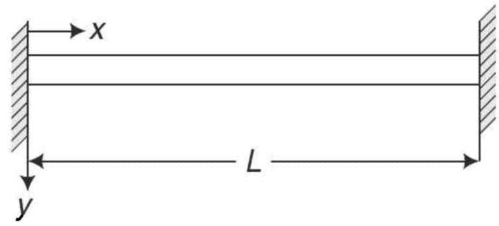

Fig. p-9.4 *Longitudinal vibration of a uniform bar*

Solution As we know, the general solution of a longitudinal vibration of a uniform bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin\frac{Z_n}{a}x + B\cos\frac{Z_n}{a}x\right)(C\sin Z_n t + D\cos Z_n t)$$

The boundary conditions for the above particular system (fixed at both ends) are

$$(u)_{x=0} = (u)_{x=1} = 0$$

The displacements of this bar at its ends are equal to zero.

Substituting these boundary conditions into the general solution, we have

$$(u)_{x=0} = \sum_{n=1,2,3,...}^{\infty} T_n\left[C\cos\left(\frac{Z_n}{a}\right)x + D\sin\left(\frac{Z_n}{a}\right)x\right] = 0 \text{ or } C = 0$$

$$(u)_{x=L} = \sum_{n=1,2,3,...}^{\infty} T_n\left[D\sin\left(\frac{Z_n}{a}\right)x\right] = 0 \text{ or } \sin\left(\frac{Z_n}{a}L\right) = 0 \text{ and } Z_n = \frac{n\pi a}{L}, \text{ where}$$

$n = 1,2,3, ...$

Hence, the normal function is $X_n(x) = D\sin\frac{n\pi x}{L}$, $n = 1, 2, 3...$

EXAMPLE. 9.5

A bar of length '*L*' fixed at one end and connected at the other end by a spring of stiffness '*k*' is as shown in Fig. p-9.5. Derive a suitable expression of motion for longitudinal vibrations.

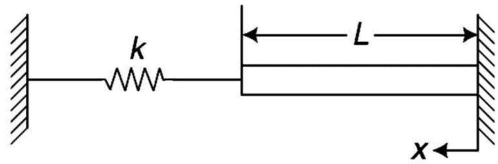

Fig. p-9.5 *Bar fixed at one end and connected at the other end by a spring*

Solution As we know, the general solution of longitudinal vibration of the bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin\frac{Z_n}{a}x + B\cos\frac{Z_n}{a}x\right)(C\sin Z_n t + D\cos Z_n t) \quad ...9.25$$

The general solution of longitudinal vibration of a uniform bar whose one end is fixed and other end is free can be written as (similar to Example 9.2).

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{nx\pi}{2L} \left(C \sin \frac{nx\pi}{2L} t + D \cos \frac{nx\pi}{2L} t \right)$$

The boundary conditions for the above particular system are

$$(u)_{x=0} = 0 \quad AE = \frac{\partial u}{\partial x} (L, t) = ku(L, t) \text{ (Tensile force = Spring force)}$$

Applying the second boundary conditions, we get

$$AE \frac{Z_n}{a} \cos \frac{Z_n}{a} L \left(C \sin \frac{n\pi a}{2L} t + D \cos \frac{n\pi a}{2L} t \right) = k \sin \frac{Z_n}{a} L \left(C \sin \frac{n\pi a}{2L} t + D \cos \frac{n\pi a}{2L} t \right)$$

$\tan \frac{Z_n L}{a} = \frac{AE}{k} \frac{Z_n}{a}$ is the required equation.

Q. Derive the differential equation of Torsional vibrations of uniform shaft or rod .

Ans :

The equation of motion for the torsional vibration of the circular uniform shafts are same as the longitudinal vibration of the uniform bars discussed in one-dimensional wave equation for lateral vibration of string Eq. 9.3.

Also the method of derivation of these equations is same as that of longitudinal vibration of bars in Eq. 9.24.

Let us consider a prismatic shaft of length 'L' subjected to torsional vibration as shown in Fig. 9.3(a).

Let us consider a small elemental length of rod 'dx' at a distance 'x' from left end, let 'T' be the applied torque and 'θ' be the angle of twist at left side of small elemental length of rod, $\left(\theta + \frac{\partial \theta}{\partial x} dx \right)$. Twist at a distance $x + dx$ from right side due to a applied torque $\left(T + \frac{\partial T}{\partial x} dx \right)$, as in Fig. 9.3(b), similar to the longitudinal vibration of rods.

J = Polar moment of inertia of shaft per unit length

I = Mass moment of inertia

G = Modulus of rigidity of the shaft material

d = Diameter of the shaft

ρ = Mass density of the material = (Mass × Volume)

From Newton's second law of motion,

Applied torque (T) = Inertia force × Angular acceleration or $T = I \times \omega$

$$\text{Net torque} = \left(T + \frac{\partial T}{\partial x} dx \right) - T = I \times \frac{d^2\theta}{dt^2} \text{ or } \left(\frac{\partial T}{\partial x} dx \right) = I \times \frac{d^2\theta}{dt^2} \qquad \text{...9.28}$$

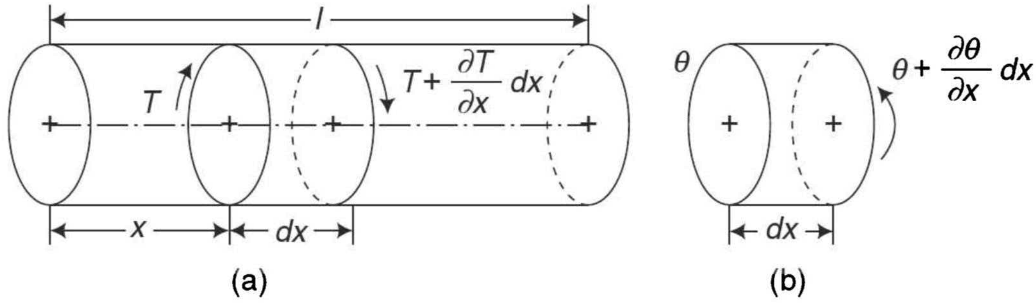

Fig. 9.3 *Torsional vibration of uniform shaft*

From mechanics of materials, the elementary torsion theory equation $\frac{T}{J}=\frac{G\theta}{l}$.

But $\frac{\theta}{l}=\frac{d\theta}{dx}$ = Twist per unit length or the rate of twist of small elemental length of rod dx.

$$\frac{T}{J}=G\frac{d\theta}{dx},\quad T=GJ\frac{d\theta}{dx},\left(\frac{\partial T}{\partial x}dx\right)=GJ\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx \qquad ...9.29$$

Comparing Eq. 9.28 and Eq. 9.29, we have

$$GJ\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx=I.\frac{d^2\theta}{dt^2} \qquad ...9.30$$

For a shaft of constant cross-section, 'GJ' is constant

$$J=\frac{\pi}{32}d^4,\ I=\frac{\pi}{32}d^4\ p.dx \text{ (mass moment of inertia)}$$

Substitute the values of 'I' and 'J' in Eq. 9.30. we get,

$$\frac{G}{\rho}\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2\theta}{\partial t^2},\ \frac{\partial^2\theta}{\partial x^2}(x,t)=\frac{1}{a^2}\frac{\partial^2\theta}{\partial t^2}(x,t) \quad \text{where } a^2=G/\rho$$

This is wave equation identical to Eq. 9.3.

$$\frac{\partial^2 y}{\partial t^2}=\frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad ...9.31$$

The general solution of the above equation can be written as

$$\theta(x,t)=\sum_{n=1}^{\infty}\left(A\sin\frac{Z_n x}{a}+B\cos\frac{Z_n x}{a}\right)(C\sin Z_n t+D\cos Z_n t). \qquad ...9.32$$

EXAMPLE 9.6

Derive the frequency equation of torsional vibrations for a free–free shaft of length 'l'.

Solution The general solution for equation of torsional vibrations for shaft can be written as Eq. 9.32:

$$\theta(x,t)=\sum_{n=1}^{\infty}\left(A\sin\frac{Z_n x}{a}+B\cos\frac{Z_n x}{a}\right)(C\sin Z_n t+D\cos Z_n t).$$

The boundary conditions for above particular system are

$\frac{\partial \theta}{\partial x}(0 \cdot t) = 0$ (strain in zero at both ends), $\frac{\partial \theta}{\partial x}(L \cdot t) = 0$

Applying the above two boundary conditions to the general solution of torsional vibrations, we get

$$\frac{\partial \theta}{\partial x} = \left(A \cos \frac{Z_n}{a} x \frac{Z_n}{a} - B \sin \frac{Z_n}{a} x \frac{Z_n}{a}\right)(C \sin Z_n t + D \cos Z_n t) \text{ or}$$

$$\frac{\partial \theta}{\partial x} = \frac{Z_n}{a}\left(A \cos \frac{Z_n}{a} x - B \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

$$\frac{\partial \theta}{\partial x} = 0 \text{ at } x = 0$$

$$\therefore \qquad A = 0, \text{ and } \frac{\partial \theta}{\partial x} = 0 \text{ at } x = l$$

$$0 = B \frac{Z_n}{a} \sin \frac{Z_n}{a} L\,(C \sin Z_n t + D \cos Z_n t)$$

$$\sin \frac{Z_n}{a} l = \sin n\pi, \quad Z_n = \frac{n\pi a}{L} \text{ or } 2\pi, f_n = \frac{n\pi a}{L}, f_n = \frac{n}{2L}\sqrt{\frac{G}{\rho}}$$

where $a = \sqrt{\frac{G}{\rho}}$ and $n = 1, 2, 3, \ldots$

The general solution can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} \cos \frac{n\pi x}{aL}\left(C \sin \frac{n\pi at}{L} + D \cos \frac{n\pi at}{L}\right)$$

EXAMPLE 9.7

A uniform shaft of length 'L' fixed at one end and free at the other end is as shown in Fig. p-9.7. Determine the free torsional vibration of the shaft.

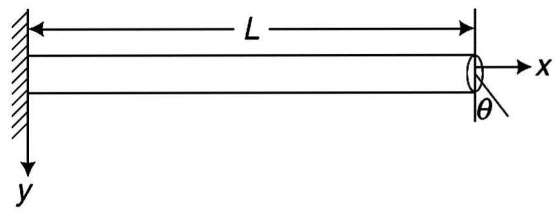

Fig. p-9.7 *Uniform shaft fixed at one end and free at the other end*

Solution The differential equation of motion for free torsional vibration of a shaft is given by $\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial \theta}{\partial x^2}$

where θ = Angular displacement, $a^2 = G/\rho$ and Z_n = Natural frequencies of the shaft ($Z_n = 2\pi f_n$)

The general solution for equation of torsional vibrations for a shaft can be written as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos \frac{Z_n}{a} x + D_n \sin \frac{Z_n}{a} x\right)$$

The boundary conditions for the above particular system are

At $\qquad x = 0, \quad \theta(0, t) = 0$

At $x = L, \quad GI_p(\partial\theta/\partial x) = 0$

where I_p is the polar moment of inertia of the shaft

Using the first boundary conditions, we get

$$\theta(0, t) = \sum_{t=1,2,3,...}^{\infty} C_n(A_n \cos Z_n t + B_n \sin Z_n t) = 0 \text{ or } C_n = 0.$$

And from second boundary condition, we get

$$\theta(x, t) = \sum_{n=1,2,3,...}^{\infty} \left(\sin \frac{Z_n}{a} x\right)(A_n \cos Z_n t + B_n \sin Z_n t)$$

$$(\partial\theta/\partial x)_{x=L} = \sum_{n=1,2,3,...}^{\infty} \frac{Zn}{a}\left(\cos \frac{Z_n}{a} L\right)(A_n \cos Z_n t + B_n \sin Z_n t) = 0$$

$$\cos \frac{Z_n}{a} L = 0, Z_n = \frac{n\pi a}{2L}, \quad \text{where } n = 1, 3, 5, ...$$

Hence, the torsional vibration of the shaft is

$$\theta(x, t) = \sum_{n=1,3,...}^{\infty} \sin \frac{n\pi x}{2L}\left(A_n \cos \frac{n\pi at}{2L} + B_n \sin \frac{n\pi at}{2L}\right)$$

where 'A_n' and 'B_n' are constants determined by initial conditions of the problem.

EXAMPLE 9.8

Derive the frequency equation for the torsional vibration of a uniform circular shaft with rotors attached rigidly at the ends as shown in Fig. p-9.8.

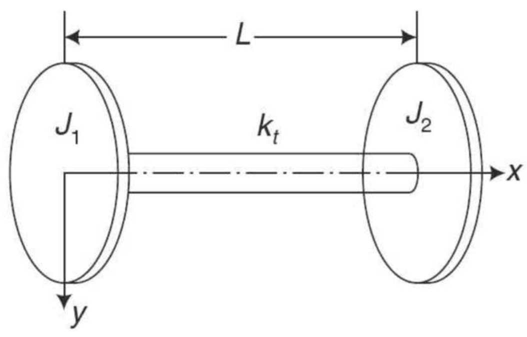

Fig. p-9.8 *Two-rotor system*

Solution The general solution for the torsional vibration of circular shafts can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,...}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos \frac{Z_n}{a} x + D_n \sin \frac{Z_n}{a} x\right)$$

where $a^2 = G/\rho$ and Z_n = Natural frequencies

($\because$ the equations of motion for torsional vibration of a circular shaft and for longitudinal vibration of uniform bars are identical)

The twisting of the shaft at both ends is produced by the inertia forces of the rotors.

The boundary conditions for the above particular system are

At $x = 0,\quad J_1(\partial^2\theta/\partial t^2) = GI_p(\partial\theta/\partial x)$

At $x = L,\quad J_2(\partial^2\theta/\partial t^2) = -GI_p(\partial\theta/\partial x)$

where G = Shear modulus of elasticity, I_p = Polar moment of inertia.

From first boundary condition, we get

$$Z_n^2 J_1 C_n + (Z_n GI_p/a)\, D_n = 0$$

And from second boundary condition we get

$$\left(Z_n^2 J_2 \cos Z_n L/a + \frac{Z_n GI_p}{a}\sin Z_n L/a\right) C_n +$$
$$\left(Z_n^2 J_2 \sin Z_n L/a - \frac{Z_n GI_p}{a}\cos Z_n L/a\right) D_n = 0$$

The frequency equation obtained by equating to zero the determinant of the coefficients of 'C_n' and 'D_n' is

$$Z_n^2\left(\cos Z_n L/a - \frac{Z_n a J_1}{GI_p}\sin Z_n L/a\right) J_2 + \frac{Z_n GI_p}{a}\left(\sin Z_n L/a + \frac{Z_n a J_1}{GI_p}\cos Z_n L/a\right) = 0$$

EXAMPLE 9.9

A pulley of moment of inertia '*J*' is rigidly attached to the free end of a uniform shaft of length '*L*' as shown in Fig. p-9.9. Determine the frequency equation for torsional vibration.

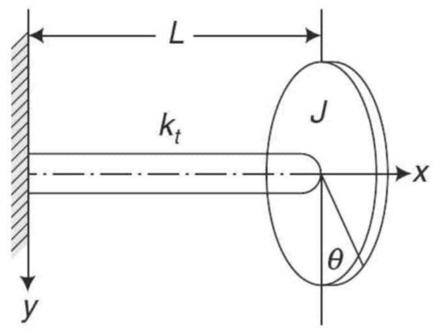

Fig. p-9.9 *Uniform shaft and pulley*

Solution The differential equation of motion for torsional vibration of the shaft are given by

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2} \qquad \text{... 9.33(a)}$$

The general solution for the torsional vibration of circular shafts can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos\frac{Z_n}{a}x + D_n \sin\frac{Z_n}{a}x\right) \qquad \text{... 9.33(b)}$$

where $a^2 = G/\rho$ and Z_n = Natural frequencies

The boundary conditions for the above particular system are

$$\theta(0, t) = 0,\ -GI_p(\partial\theta/\partial x)_{x=L} = J(\partial^2\theta/\partial t^2)$$

i.e. the angular displacement of the shaft at the fixed end is equal to zero and the restoring torque of the shaft at the free end is equal to the inertia moment of the pulley.

From first boundary condition $C_n = 0$; and from secondary boundary condition,

$-\dfrac{GI_p Z_n}{a}\cos\dfrac{Z_n L}{a} = -JZ_n^2 \sin\dfrac{Z_n L}{a}$ or $-\tan\dfrac{Z_n L}{a} = \dfrac{GI_p}{aJZ_n}$, which is the frequency equation.

Derive the differential equation of Transverse vibration of a beam

Ans :

Let us consider a simply supported beam of uniform cross section subjected to transverse vibration as shown in Fig. 9.4.

Assumption made while deriving the expression for transverse vibration of beams are the following:

1. The deformation of the beam is assumed due to moment and shear force.
2. There are no axial forces acting on the beam and effects of shear deflection are neglected.

We know from mechanics of materials, the differential equation of motion for the transverse vibration of beam, the deflection curve of a beam is given by

$$EI\frac{d^2y}{dx^2} = -M \quad \text{... 9.34}$$

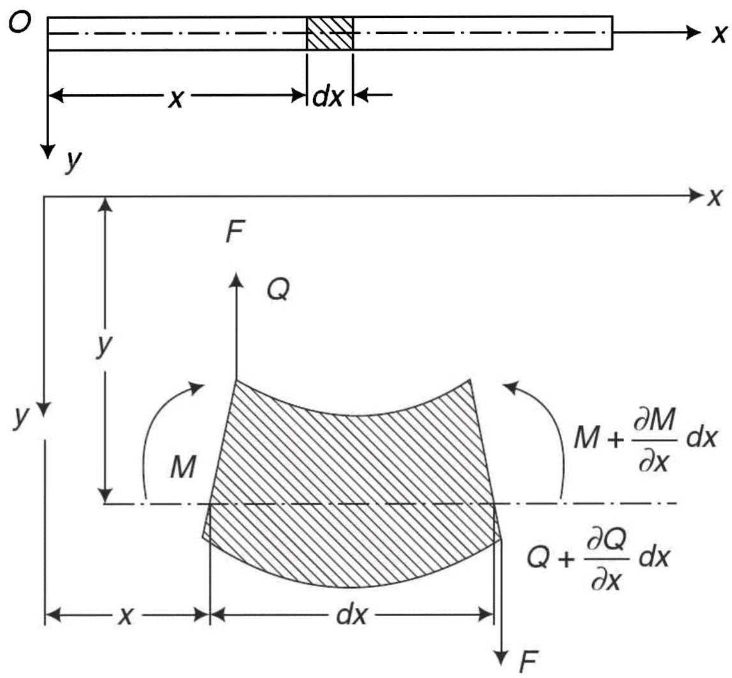

Fig. 9.4 *Transverse vibration of beams*

where y = Deflection of the beam, M = Bending moment at any cross-section

EI = known as the Flexural rigidity of the beam and is assumed as a constant.

Differentiating Eq. 9.34 twice, we get

$$EI\frac{d^3y}{dx^3} = -F \quad \text{...9.35}$$

$$EI\frac{d^4y}{dx^4} = W \quad \text{...9.36}$$

where F = Shear force, W = Intensity of loading.

(As we know the relationship between the shear force 'F', the intensity of loading 'W' and bending moment 'M').

In case of free transverse vibration of beams without application of external loading, it is very important to consider the inertia forces $\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}$ as the loading intensity along the entire length of beam. Then Eq. 9.36 becomes

$$EI\frac{\partial^4 y}{\partial x^4} = -\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2} \quad ...9.37$$

Here, partial derivatives are used because of the deflection of the beam 'y' is a function of 'x' and 't'.

$$\frac{EIg}{\rho A}\frac{\partial^4 y}{\partial x^4} = -\frac{\partial^2 y}{\partial t^2}$$

Let $$a^2 = \frac{EIg}{\rho A}$$

$\therefore$ $$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \quad ...9.38$$

is the differential equation of motion for the transverse vibration for a simply supported beam of uniform cross-section including transverse inertia and stiffness of the beam.

The general solution for transverse vibration of beams is given by the expression

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx. \quad ...9.39$$

EXAMPLE 9.10

A uniform beam fixed at one end and simply supported at the other end is having transverse vibrations. Derive a suitable expression for frequency.

Solution The general solution for transverse vibration is given by the expression 9.39.

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$$

The boundary conditions for the above particular system are

$$\left.\begin{array}{l} y(0, t) = 0 \\ \frac{dy}{dx}(0, t) = 0 \end{array}\right\} \text{for fixed end,} \quad \left.\begin{array}{l} y(L, t) = 0 \\ \frac{d^2 y}{dx^2}(L, t) = 0 \end{array}\right\} \text{for simply supported end}$$

Applying the above boundary conditions for the general solution of transverse vibration is given by expression,

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$$

We get $y(0, t) = A + C = 0$

Differentiating the above equation w.r.t. x,

$$\frac{dy}{dx}(x, t) = c\,[A \sin h\, cx + B \cos h\, cx - C \sin cx + D \cos cx\,]$$

Differentiating again the above equation w.r.t. x,

$$\frac{d^2y}{dx^2}(x, t) = c^2\,[A \cos h\, cx + B \sin h\, cx - C \cos cx - D \sin cx]$$

$$\therefore \qquad \frac{dy}{dx}(0, t) = B + D = 0$$

$$y(L, t) = A\,(\cos h\, cL - \cos cL) + B\,(\sin h\, cL - \sin cL) = 0 \text{ and}$$

$$\frac{d^2y}{dx^2}(L, t) = c^2\,[A \cos h\, cL + B \sin h\, cL - C \cos cL - D \sin cL] = 0$$

$$A\,(\cos h\, cl + \cos cl\,) + B\,(\sin h\, cl + \sin cl) = 0$$

$$A\,(\cos h\, cl - \cos cl\,) + B\,(\sin h\, cl - \sin cl) = 0$$

$$A\,(\cos h\, cl + \cos cl) + B\,(\sin h\, cl + \sin cl) = 0$$

Eliminating 'A' and 'B' from the above two equations, we get

$$(\cos h\, cl - \cos cl\,)\,(\sin h\, cl + \sin cl) - (\sin h\, cl - \sin cl)\,(\cos h\, cl + \cos cl) = 0$$

Solving it, we get frequency equations as

$$\cos cl \sin h\, cl - \sin cl \cos h\, cl = 0$$

$$\tan cl = \tan h\, cl$$

EXAMPLE 9.11

Find frequency equation of a uniform beam fixed at one end and free at the other end for transverse vibrations.

Solution The general solution for transverse vibration is given by the expression 9.39

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx.$$

The boundary conditions for the above particular system are

$$y(0, t) = 0 \text{ (zero deflection at fixed end)}, \quad \frac{dy}{dx}(0, t) = 0 \text{ (zero slope)}$$

$$\frac{d^2y}{dx^2}(L, t) = 0 \text{ (zero bending moment)}, \quad \frac{d^3y}{dx^3}(L, t) = 0 \text{ (zero shear force)}$$

Applying boundary conditions, we get $0 = A + C$, $A = -C$

$$\frac{dy}{dx}(x, t) = c\,(A \sin h\, cx + B \cos h\, cx - C \sin cx + D \cos cx) = 0$$

$$\frac{dy}{dx}(0, t) = 0 = B + D \qquad \therefore\ B = -D$$

$$\frac{d^2y}{dx^2}(L, t) = c^2 [A (\cos h\, cL + \cos cL) + B (\sin h\, cL + \sin cL)] = 0$$

$$\frac{d^3y}{dx^3}(L, t) = c^3 [A (\sin h\, cL - \sin cL) + B (\cos h\, cL + \cos cL)] = 0$$

$$[\cos h\, cL + \cos cL]^2 - (\sin h^2\, cL - \sin^2 cL) = 0$$

$$\cos h^2\, cL + \cos^2 cL + 2 \cos h\, cL \cos cL - \sin h^2\, cL + \sin^2 cL = 0$$

Solving, we get

$$\cos h\, cL \cos cL + 1 = 0$$

The above equation can be solved for cL to find natural frequency of the system.

EXAMPLE 9.12

Derive frequency equation for a beam with both ends free and having transverse vibrations.

Solution The general solution for transverse vibration is given by the expression 9.39

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx, \text{ where } c^2 = Z_n \sqrt{\frac{\rho A}{EI}}.$$

The boundary conditions for the above particular system are

$$\frac{d^2y}{dx^2}(0, t) = 0 \quad \text{(Because bending moment should be zero)}$$

$$\frac{d^2y}{dx^2}(L, t) = 0 \quad \text{(Because bending moment should be zero)}$$

$$\frac{d^3y}{dx^3}(0, t) = 0 \quad \text{(Because shear force should be zero)}$$

$$\frac{d^3y}{dx^3}(L, t) = 0 \quad \text{(Because shear force should be zero)}$$

Now applying the boundary conditions, for general solution of transverse vibration, we get

$$\frac{d^2y}{dx^2}(x, t) = c^2 [A \cos h\, cx + B \sin h\, cx - C \cos cx - D \sin cx]$$

$$\frac{d^2y}{dx^2}(0, t) = c^2 (A - C) = 0$$

$$\therefore \quad A = C$$

$$\frac{d^3y}{dx^3}(x, t) = c^3 [A \sin h\, cx + B \cos h\, cx + C \sin cx - D \cos cx]$$

$$\frac{d^3y}{dx^3}(0, t) = c^3\,[B - D] = 0$$

$$\therefore \quad B = D$$

$$\frac{d^2y}{dx^2}(L, t) = c^2\,[A\,(\cos h\; cL - \cos cL) + B\,(\sin h\; cL - \sin cL)] = 0$$

$$\frac{d^3y}{dx^3}(L, t) = c^3\,[A\,(\sin h\; cL + \sin cL) + B\,(\cos h\; cL - \cos cL)] = 0$$

$$A\,(\cos h\; cL - \cos cL) + B\,(\sin h\; cL - \sin cL) = 0$$

$$A\,(\sin h\; cL + \sin cL) + B\,(\cos h\; cL - \cos cL) = 0 \text{ or}$$

$$(\cos h\; cL - \cos cL)^2 - (\sin h^2\; cL - \sin^2 cL) = 0$$

$$\cos h^2\; cL + \cos^2 cL - 2\cos h\; cL \cos cL - \sin h^2\; cL + \sin^2 cL = 0$$

$$\cos h^2\; cL - \sin h^2\; cL = 1 \text{ and } \cos^2 cL + \sin^2 cL = 1$$

$$\cos h\; cL + \cos cL = 1$$

IMPORTANT EQUATIONS IN VIBRATIONS OF A CONTINUOUS SYSTEM

1. Lateral vibration of a string One-dimensional wave equation for lateral vibration of a string is given by the expression

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad ...9.40$$

The general solution for lateral vibration of a string is given by the expression

$$y(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{Z}{a}\right)x + B_n \sin\left(\frac{Z}{a}\right)x\right)[C_n \cos Zt + D_n \sin Zt] \qquad ...9.41$$

2. Longitudinal vibration of bars The differential equation of motion for longitudinal vibration of bars is given by the expression

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad ...9.42$$

The general solution for longitudinal vibration of bars is given by the expression

$$u(x, t) = \sum_{n=1}^{\infty}\left(A \sin\frac{Z_n}{a}x + B \cos\frac{Z_n}{a}x\right)[C \sin Z_n t + D \cos Z_n t] \qquad ...9.43$$

3. Torsional vibration of circular rods or shafts The differential equation of motion for torsional vibration of circular rods or shafts is given by the expression

$$\frac{\partial^2 \theta}{\partial t^2} = a^2\frac{\partial^2 \theta}{\partial x^2} \qquad ...9.44$$

The general solution for torsional vibration of circular rods or shafts is given by the expression

$$\theta(x, t) = \sum_{n=1}^{\infty}\left(A \sin\frac{Z_n x}{a} + B \cos\frac{Z_n x}{a}\right)(C \sin Z_n t + D \sin Z_n t) \qquad ...9.45$$

4. Transverse vibration of beams The differential equation of motion for transverse vibration of beams is given by the expression

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \qquad ...9.46$$

The general solution for transverse vibration of beams is given by the expression

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx \qquad ...9.47$$