MATHEMATICS Module: 1 3rd Semester

Anupam pradhan

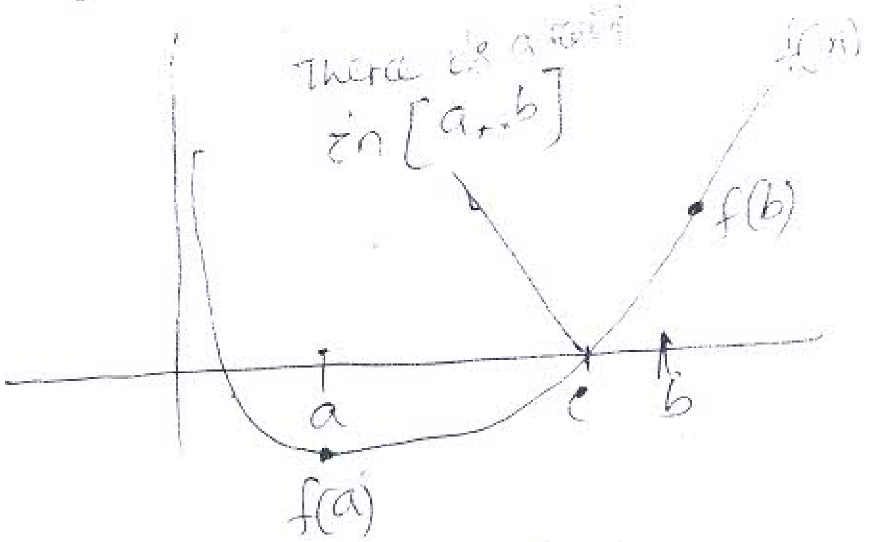
Solution of Non-linear equation in one Variable:-

→ Bisection Method or Binary search method or dichotonomy Method is based on the

Bolzano's theorem for continuous function (Bolzano):-

If a function $f(x)$ is continuous on an Interval $[a,b]$ and $f(a).f(b)<0$ then a value $c \in (a,b)$ exist for which $f(c)=0$

Graphically represent as



sign of $f(a) \neq$ sign of $f(b)$

Bisection Algorithm:-

We have to solve the equation $f(x)=0$ numerically.

Let $f(x)$ be continuous in $[a,b]$

Let a and b be two points such that $f(a)$ and $f(b)$ are of opposite signs.

Then there exists at least one root betwe[en]
a and b.
For definiteness, let $f(a)$ be negative and
$f(b)$ are of opposite signs.
Then the root lies between a and b.
Let $x_0 = \frac{a+b}{2}$ be an initial approximation
to the root. $f(x_0)$ is to be evaluated.
Now, three cases will arise.
They are either $f(x_0)=0$ or $f(x_0)<0$ or $f(x_0)>0$

Case-I:
When $f(x_0)=0$
Then x_0 is the exact root of the equation
$f(x)=0$

Case-II:
When $f(x_0)<0$
Then root lies between x_0 and b.
Let $x_1 = \frac{x_0+b}{2}$ be the next approximation
to the root.
Then $f(x_1)$ is evaluated. $f(x_1)$ may be zero
or positive or negative.
[illegible] foregoing process is repeated until
[illegible] upto desired

Case - III

When $f(x_0) > 0$

Then root lies between a and x_0

Let $x_1 = \frac{a + x_0}{2}$ be the next approximation to the root.

Then $f(x_1)$ is evaluated. $f(x_1)$ may be zero or positive or negative.

The foregoing process is repeated until we get the root correct up to desired accuracy

(Q) Find an approximation to $\sqrt{2}$ correct to two decimal places using the bisection method.

Solution:-

Let $x = \sqrt{2}$

Bothside squaring we get

$x^2 = (\sqrt{2})^2$

$\Rightarrow x^2 = 2$

$\Rightarrow x^2 - 2 = 0$

So $\sqrt{2}$ is the positive root of the equation

$x^2 - 2 = 0$

Let $f(x) = x^2 - 2$

$f(0) = (0)^2 - 2 = 0 - 2 = -2$

$f(1) = (1)^2 - 2 = 1 - 2 = -1$

$f(2) = (2)^2 - 2 = 4 - 2 = 2$

between 1 and 2.
Let the initial approximation to the root be given by $x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

$f(x_0) = f(1.5) = (1.5)^2 - 2 = 2.25 - 2 = 0.25$

So, the root lies between 1 and 1.5

$x_1 = \frac{1+1.5}{2} = 1.25$

$f(x_1) = f(1.25) = (1.25)^2 - 2$

$= 1.5625 - 2 = -0.4375$

So, the root lies between 1.25 and 1.5

$x_2 = \frac{1.25+1.5}{2} = 1.375$

$f(x_2) = f(1.375) = (1.375)^2 - 2$

$= 1.890625 - 2$

$= -0.109375$

So the root lies between 1.375 and 1.5

$x_3 = \frac{1.375+1.5}{2} = 1.4375$

$f(x_3) = f(1.4375) = (1.4375)^2 - 2$

$= 2.06640625 - 2$

$= 0.06640625$

So the root lies between 1.375 and 1.4375

$x_4 = \frac{1.375+1.4375}{2} = 1.40625$

$f(x_4) = f(1.40625) = (1.40625)^2 - 2$

So the root lies between 1.40625 and 1.4375

$$x_5 = \frac{1.40625 + 1.4375}{2} = 1.421875$$

$$f(x_5) = f(1.421875) = (1.421875)^2 - 2$$
$$= 0.021728515б$$

So the root lies between 1.40625 and 1.421875

$$x_6 = \frac{1.40625 + 1.421875}{2} = 1.4140625$$

$$f(x_6) = f(1.4140625)$$
$$= (1.4140625)^2 - 2$$
$$= -0.00042724б1$$

So the root lies between 1.4140625 and 1.421875

$$x_7 = \frac{1.4140625 + 1.421875}{2} = 1.41796875$$

So the value of $\sqrt{2}$ correct up to two decimal places is 1.41 (Ans).

(Q): Find an approximation to $\sqrt{3}$ correct to two decimal places using the bisection method.

Solution: Let $x = \sqrt{3}$

Both side squaring we get

$(x)^2 = (\sqrt{3})^2$

$\Rightarrow x^2 = 3$

$\Rightarrow x^2 - 3 = 0$ ... positive root of the

Suppose $f(x) = x^2 - 3$

$$f(0) = (0)^2 - 3 = 0 - 3 = -3$$

$$f(1) = (1)^2 - 3 = 1 - 3 = -2$$

$$f(2) = (2)^2 - 3 = 4 - 3 = 1$$

Therefore the equation $f(x) = 0$ has a root between 1 and 2.

Let the Initial approximation to the root be given by

$$x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$f(x_0) = f(1.5) = (1.5)^2 - 3 = 2.25 - 3 = -0.75 < 0$

As $f(1.5) < 0$ and $f(2) > 0$, the root lies in the Interval $(1.5, 2)$.

Therefore the second approximation x_1 is

$$x_1 = \frac{1.5 + 2}{2} = 1.75$$

Also $f(x_1) = (1.75)^2 - 3.$

$= 0.0625 > 0$

⇒ the root lies in $(1.5, 1.75)$

Hence the 3rd approximation x_2 is

$$\therefore x_2 = \frac{1.5 + 1.75}{2} = 1.625$$

But $f(x_2) = f(1.625) = (1.625)^2 - 3$

$= -0.359375 < 0$

⇒ the root lies in (x_1, x_2)

Now $f(x_3) = -0.1523437 < 0$

The root lies in (x_1, x_3); Hence

$$x_4 = \frac{1.75 + 1.6875}{2} = 1.71875$$

Again $f(x_4) = -0.0458984 < 0$

$\Rightarrow$ the root lies in (x_4, x_1)

$$x_5 = \frac{1.75 + 1.71785}{2} = 1.734375$$

Which is accurate upto 2-decimals places since $\sqrt{3} = 1.7320598$. (Ans).

(Q) Find third approximate value of root of equation $x^3 - 3x + 1 = 0$ lying between 0 and 1.

Solution:

Given that $f(x) = x^3 - 3x + 1$

$a=0 \quad b=1$

$f(0) = (0)^3 - 3.0 + 1$
$= 0 - 0 + 1 = 1 > 0$

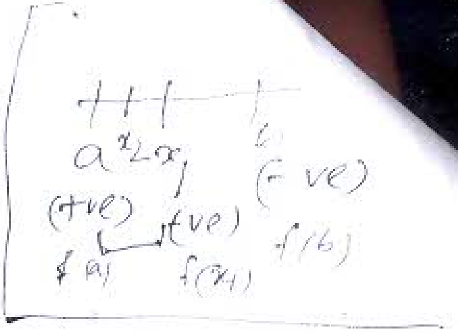
$f(1) = (1)^3 - 3\times 1 + 1$
$= 1 - 3 + 1 = -1 < 0$

As $f(0) > 0$ and $f(1) < 0$, the root lies in the Interval $(0, 1)$ the first approximation x_1 is therefore

$x_1 = \frac{0+1}{2} = \frac{1}{2}$

$= \frac{1}{8} - \frac{3}{2} + 1$

$= \frac{1-12+8}{8} = \frac{-3}{8} < 0$



The second approximation x_2 is

$$x_2 = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$$

$$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^3 - 3\cdot\frac{1}{4} + 1$$

$$= \frac{1}{64} - \frac{3}{4} + 1$$

$$= \frac{1-48+64}{64} = \frac{17}{64} > 0$$

The Third approximation x_3 is

$$x_3 = \frac{\frac{1}{2}+\frac{1}{4}}{2} = \frac{\frac{3}{4}}{2} = \frac{3}{8}$$

H.W

Q:(1) Find fourth approximate value of root of equation $x^3 - x - 1 = 0$ lying between 0 and 2.

H.W

Q:(2) perform five Iterations of the bisection method to obtain the smallest root of the equation $f(x) = x^3 - 5x + 1 = 0$

H.W

Q:(3) perform five Iterations of the bisection ... the smallest root of

Note:-

(1) Bisection-Method is also called bracketing method because by this method we bracket (or enclose) the root between the ends points of a sequence of closed intervals.
(2) It is also known as interval-halving method.

Newton-Raphson Method:-(N-R method)

We have to solve the equation $f(x)=0$ numerically.
Let $f(x)$ be continuous and differentiable in $[a,b]$
Let a and b be two points such that $f(a)$ and $f(b)$ are of opposite signs.
Then there exists at least one root between a and b.
Let $x_0 = a$ or b or $\frac{a+b}{2}$ be an approximate of $f(x)=0$ and let $x_1 = x_0+h$ be the exact root so that $f(x_1)=0$
so $f(x_0+h)=0$
expanding $f(x_0+h)$ by Taylor's series
we obtain $f(x_0)+hf'(x_0)+\frac{h^2}{\underline{|2}} f''(x_0)+\cdots=0$

Neglecting 2nd order and higher order of derivative

$f(x_0)+hf'(x_0)=0$

Which gives $h = -\frac{f(x_0)}{f'(x_0)}$

... approximation than x_0 is there

$f(x_0)$

Successive approximations $x_2, x_3 \ldots x_{n+1}$ can similarly be obtained as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

. . . .

. . . .

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots}$$ is the general

form of Newton-Raphson method

(Q) Use the Newton-Raphson method to find a root of the equation

$f(x) = x^3 - 2x - 5$

Solution:

$f(x) = x^3 - 2x - 5$

$f(2) = (2)^3 - 2\times2 - 5 = 8 - 4 - 5 = 8 - 9 = -1 < 0$

$f(3) = (3)^3 - 2\times3 - 5 = 27 - 6 - 5 = 16 > 0$

$f'(x) = 3x^2 - 2$

The Formula for N-R method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Hence it gives $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$

$$x_{n+1} = \frac{3x_n^3 - 2x_n - x_n^3 + 2x_n + 5}{3x_n^2 - 2}$$

choose $x_0 = 2$ put $n=0$

$$x_1 = \frac{2x_0^3+5}{3x_0^2-2} = \frac{2(2)^3+5}{3(2)^2-2}$$

$$= \frac{2\cdot 8+5}{3\cdot 4-2} = \frac{16+5}{12-2} = \frac{21}{10} = 2\cdot 1$$

$f(2.1) = (2\cdot 1)^3 - 2\times(2\cdot 1) - 5$
$= 11\cdot 23$

put $n=1$

$$x_2 = \frac{2x_1^3+5}{3x_1^2-2} = \frac{2\times(2\cdot 1)^3+5}{3\times(2\cdot 1)^2-2} = 2.095$$

Hence the correct root is 2.095

(Q) Find the smallest positive root of $x^3-3x-5=0$ correct up to three decimal places by Newton-Raphson method.

Solution:

Here $f(x) = x^3-3x-5$
and $f'(x) = 3x^2-3$

$f(1) = (1)^3 - 3\times 1 - 5 = 1-3-5 = -7$

$f(2) = (2)^3 - 3\times 2 - 5 = 8-6-5 = -3$

$f(3) = (3)^3 - 3\times 3 - 5 = 27-9-5 = 13$

... real root lies between 2 and 3

By Newton-Raphson iterative formula we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3} \quad \text{—— (1)}$$

Initial approximation of the root could be taken anywhere between 2 and 3.
Let us choose $x_0 = 3$
and put $n = 0$ in (1) then

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{x_0^3 - 3x_0 - 5}{3x_0^2 - 3}$$

$$= 3 - \frac{(3)^3 - 3\times 3 - 5}{3\times(3)^2 - 3} = 3 - \frac{27 - 9 - 5}{27 - 3}$$

$$= 3 - \frac{13}{24}$$

$$= \frac{72 - 13}{24} = \frac{59}{24} = 2.46.$$

Now taking $n = 1$ in (1) we get

$$x_2 = x_1 - \frac{x_1^3 - 3x_1 - 5}{3x_1^2 - 3} = x_1 - \frac{x_1^3 - 3x_1 - 5}{3(x_1^2 - 1)}$$

$= 2.46 - \dfrac{(2.46)^3 - 3\times 2.46 - 5}{3((2.46)^2 - 1)}$

$= 2.295.$

Similarly $x_3 = x_2 - \dfrac{x_2^3 - 3.x_2 - 5}{3(x_2^2 - 1)}$

$= 2.295 - \dfrac{(2.295)^3 - 3\times 2.295 - 5}{3((2.295)^2 - 1)}$

$= 2.279.$

and $x_4 = x_3 - \dfrac{x_3^3 - 3x_3 - 5}{3(x_3^2 - 1)}$

$= 2.279 - \dfrac{(2.279)^3 - 3\times 2.279 - 5}{3((2.279)^2 - 1)} = 2.279$

Thus the root can be taken as 2.279 correct to three places of decimal.

(Q): Finding the square root of 3 using the Newton-Raphson method. (correct to two decimal places)

Solution:

The given equation is $x^2 - 3 = 0$

Here $f(x) = x^2 - 3$

$\Rightarrow f'(x) = 2x$

$f(1) = (1)^2 - 3 = 1 - 3 = -2$

$f(2) = (2)^2 - 3 = 4 - 3 = 1$

∴ root lies between 1 and 2.

Newton Raphson's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + 3}{2x_n} = \frac{x_n^2 + 3}{2x_n} \longleftarrow (1)$$

Put $n=0$ in eqn (I) we get

$$x_{0+1} = \frac{x_0^2 + 3}{2x_0} = \frac{(1)^2 + 3}{2 \times 1} = \frac{1+3}{2} = \frac{4}{2} = 2$$

$x_1 =$

Put $n = 1$ in eqn (I) we get

$$x_2 = \frac{x_1^2 + 3}{2 \times x_1} = \frac{(2)^2 + 3}{2 \times 2} = \frac{4+3}{4} = \frac{7}{4} = 1.75$$

Put $n = 2$ in eqn (I) we get

$$x_3 = \frac{(x_2)^2 + 3}{2 \times x_2} = \frac{(1.75)^2 + 3}{2 \times 1.75} = \frac{3.0625 + 3}{3.5}$$

$$x_3 = \frac{6.0625}{3.5} = 1.7321$$

Which is accurate up to 2-decimals places since $\sqrt{3} \cong 1.73$. (Ans)

H.W
(Q) Using N-R Method solve the equation $x^2 - 2 = 0$ numerically choosing the initial guess $x_0 = 1$. Ans: 1.41

H.W
(Q) Find the smallest positive root of $x^3 - 5x + 3 = 0$ correct up to three decimal places by N-R Method. Ans: 0.656

H.W
(Q) Using N-R Method solve the equation $x \sin x + \cos x = 0$ Ans: 2.79

Secant Method:

We have to solve the equation $f(x) = 0$ numerically.
Let $f(x)$ be continuous in $[a, b]$
Let a and b be two points such that $f(a)$ and $f(b)$ are of opposite signs.
So, root lies between a and b.
Let $x_0 = a$ and $x_1 = b$ be two initial approximations to the exact root "ξ" of the equation $f(x) = 0$
Then $f(x_0)$ and $f(x_1)$ can be calculated.
So, the next approximation to the root is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{\{f(x_1) - f(x_0)\}} \times f(x_1)$$

The successive approximations can be obtained as follows.

$$x_3 = x_2 - \frac{x_2 - x_1}{\{f(x_2) - f(x_1)\}} \times f(x_2)$$

$$x_4 = x_3 - \frac{x_3 - x_2}{\{f(x_3) - f(x_2)\}} \times f(x_3)$$

$\vdots$

In general

$$\boxed{x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\{f(x_k) - f(x_{k-1})\}} \times f(x_k)}$$

for $k = 1, 2, 3 \ldots$

(Q) Find two iterations by secant method to obtain an approximation to a $x^3 - x - 1 = 0$ starting with $x_0 = 1$ and $x_1 = 2$.

Solution:

Given that $f(x) = x^3 - x - 1$

$f(1) = (1)^3 - 1 - 1 = 1 - 2 = -1 < 0$

and $f(2) = (2)^3 - 2 - 1 = 8 - 3 = 5 > 0$

The roots lies between 1 and 2.

Given that $x_0 = 1$, $x_1 = 2$

Secant Iterative method is

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\{f(x_k) - f(x_{k-1})\}} \times f(x_k) \quad \text{--- (1)}$$

put $k = 1$ in eqn (1) we get

$= 2 - \frac{2-1}{5+1} \times 5$

$= 2 - \frac{5}{6} = \frac{12-5}{6} = \frac{7}{6} = 1.16$

put $k = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} \times f(x_2)$$

$$x_3 = 1.16 - \frac{1.16 - 5}{-0.578 - 5} \times (-0.578) = 1.41$$

(Q) Determine the root of the equation $\cos x - xe^x = 0$ using the secant method correct to four places of decimal.

solution:

let $f(x) = \cos x - xe^x = 0$

and let us take the two initial approximations as $x_0 = 0$ and $x_1 = 1$

There fore $f(x_0) = f(0) = 1 > 0$

$f(x_1) = f(1) = \cos 1 - e = 0.99 - \cancel{2.718} = \cancel{-2.177}$ $\ .98$ $-1.718 < 0$

Now by Secant method we have

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\{f(x_k) - f(x_{k-1})\}} \times f(x_k) \quad \text{——} (1)$$

put $k = 1$ in eqn(1) we get

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_1)$$

$$= 1 - \frac{1-0}{\cancel{-1.77798} - 1 \; -1.718 - 1} \times (\cancel{-2.17798})(-1.718)$$

$$= 0.36799$$

$$f(x_2) = \cos(0.36799) - 0.36799\, e^{(0.36799)} = \cancel{0.51987} \; 0.4688$$

Put $k = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} \times f(x_2)$$

$$= 0.36799 - \frac{0.36799 - 1}{0.46889 - (-1.77899)} \times (0.46889)$$

$$= 0.54603.$$

$$f(x_3) = \cos(0.54673) - (0.54673)\, e^{0.54673}$$

$$= 0.18729$$

Put $k = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} \times f(x_3) = 0.53171$$

Repeating the process the successive approximations are $x_5 = 0.56690$, $x_6 = 0.56799$,

$x_7 = 0.56714$

Hence the required root is 0.5671 correct to four decimal places.

H.W

(Q) Find two iterations by secant method to obtain an approximation to a $x^3-2x-1=0$. Starting with $x_0=1$ and $x_1=2$.

H.W

(Q) Using the secant method solve the equation $x^2-2=0$.

H.W

(Q) Find two iterations by secant method to obtain an approximations to a $x-2\sin x=0$. Starting with $x_0=\frac{\pi}{2}$, $x_1=\pi$.

The direct Iteration method (or Fixed point Iteration)

We have to solve the equation $f(x)=0$ numerically.

let $f(x)$ be continuous in $[a,b]$

let a and b be two points such that $f(a)$ and $f(b)$ are of opposite sign

let x_0 be an approximation to the root of the equation $f(x)=0$

The equation $f(x)=0$ can be rewritten in the form

$$x=\phi(x) \quad \text{——— (1)}$$

Substituting x_0 for x on the right side of equation (3) we obtain the first approximation as $x_1=\phi(x_0)$

The successive approximations are given by

$$x_2=\phi(x_1)$$

In general, $\boxed{x_{k+1} = \phi(x_k)}$ for $k = 0, 1, 2, 3 \ldots$

The numbers $x_1, x_2, x_3 \ldots$ are called iterates.

Convergence of Fixed-Point iteration:

Let "ξ" be an exact root of the equation $f(x) = 0$ and the point "ξ" is contained in the Interval $[a, b]$. Let $\phi(x)$ and $\phi'(x)$ be continuous in $[a, b]$ where $f(x) = 0$ has been re-written in the form $x = \phi(x)$. If there is a proper fraction "l" such that

$|\phi'(x)| \leq l < 1$ for all values of x in $[a, b]$

then the sequence of approximations $x_0, x_1, x_2 \ldots$ defined by $x_{k+1} = \phi(x_k)$ for $k = 0, 1, 2, 3 \ldots$ converges to the root "ξ" provided that the Initial approximation x_0 is choosen in $[a, b]$.

(Q) Find the first three iterates starting from $x_0 = 0$ for the iteration function $\phi(x) = x^2 + 1$.

Solution:

Given that function $\phi(x) = x^2 + 1$

Given that $x_0 = 0$

By Fixed Iterative formula is

$$x_{k+1} = \phi(x_k) \quad \text{——— (1)}$$

put $k=0$ in eq^n(1) we get

$x_{0+1} = \phi(x_0)$

$x_1 = \phi(x_0)$

$x_1 = x_0^2+1 = (0)^2+1 = 0+1 = 1$

put $k=1$

$x_{1+1} = \phi(x_1)$

$x_2 = x_1^2+1 = (1)^2+1 = 1+1 = 2$

put $k=2$

$x_{2+1} = \phi(x_2)$

$x_3 = x_2^2+1 = (2)^2+1 = 4+1 = 5$

(Q) Find the first three iterates starting from $x_0 = \frac{1}{2}$ for the iteration function $\phi(x) = x^2$.

Solution:

Given that function $\phi(x) = x^2$

Given that $x_0 = \frac{1}{2}$

By fixed Iterative formula is

$x_{k+1} = \phi(x_k)$ ——— (1)

put $k=0$ in eq^n(1) we get

$x_{0+1} = \phi(x_0)$

$x_1 = \phi(x_0) = (\frac{1}{2})^2 = \frac{1}{4}$

put $k=1$

$x_{1+1} = \phi(x_1)$

$x_2 = (\frac{1}{4})^2 = \frac{1}{16}$

(Q) Finding the square root of 5 using fixed point iteration method. (Correct up to two decimal places)

Solution:

Given that $x = \sqrt{5}$

Squaring on both side we get

$x^2 = 5$

$x^2 - 5 = 0$

Dividing by x on both side

$\frac{x^2-5}{x} = \frac{0}{x} \Rightarrow \frac{x^2-5}{x} = 0$

$\Rightarrow \frac{x^2}{x} - \frac{5}{x} = 0$

$\Rightarrow x - \frac{5}{x} = 0$

$\Rightarrow x = \frac{5}{x}$

Adding "x" on both side we get

$\Rightarrow x + x = \frac{5}{x} + x$

$\Rightarrow 2x = \frac{5}{x} + x$

$\Rightarrow x = \frac{1}{2}\left(\frac{5}{x} + x\right)$

which is of the form $x = \phi(x)$

where $\phi(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$

Here $f(x) = x^2 - 5$

$f(2) = (2)^2 - 5 = 4 - 5 = -1 < 0$

$f(3) = (3)^2 - 5 = 9 - 5 = 4 > 0$

As $f(2) < 0$ and $f(3) > 0$ then root lies between 2 and 3

Hence $\phi'(x) = \frac{1}{2}\left(\frac{-5}{x^2} + 1\right)$

clearly $|\phi'(x)| < 1$ for all x near 2.

Here $x_0 = 2$ then $x_1 = \phi(x_0)$ (put $K = 0$)

$x_1 = \frac{1}{2}\left(\frac{5}{x_0} + x_0\right) = \frac{1}{2}\left(\frac{5}{2} + 2\right) = \frac{1}{2}\left(\frac{5+4}{2}\right)$

$\Rightarrow x_1 = \frac{1}{2}\left(\frac{9}{2}\right) = \frac{9}{4} = 2.25$

put $K = 1$

$x_2 = \phi(x_1) = \frac{1}{2}\left(\frac{5}{2.25} + 2.25\right) = 2.236$

put $K = 2$

$x_3 = \phi(x_2) = \frac{1}{2}\left(\frac{5}{2.236} + 2.236\right) = 2.23607$

So the root correct up to 2-decimal places is 2.23.

(Q) Find a real root of the equation $x^3 + x^2 - 1 = 0$ correct upto two decimal places by fixed point Iteration method. Show that the Iteration function $\phi(x) = \frac{1}{\sqrt{x+1}}$ satisfies the condition for the Iteration scheme to converge to a real root lying between 0 and 1.

Solution:-

The given equation is $x^3 + x^2 - 1 = 0$

Here $f(x) = x^3 + x^2 - 1$

$f(0) = (0)^3 + (0)^2 - 1 = 0 - 1 = -1 < 0$

$f(1) = (1)^3 + (1)^2 - 1 = 1 + 1 - 1 = 2 - 1 = 1 > 0$

So, a real root lies between 0 and 1.

$x^3 + x^2 - 1 = 0$

$$\Rightarrow x^2 = \frac{1}{(x+1)}$$

$$\Rightarrow x = \frac{1}{\sqrt{x+1}}$$

$$\Rightarrow x = \phi(x) \text{ where } \phi(x) = \frac{1}{\sqrt{x+1}}$$

$$\phi'(x) = \frac{-1}{2(x+1)^{3/2}}$$

clearly $|\phi'(x)| < 1 \ \forall\ x \in (0,1)$

so $\phi(x) = \frac{1}{\sqrt{x+1}}$ satisfies the condition for the Iteration scheme to converge a real root lying between 0 and 1

$x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$

Let $x_0 = 0.5$

By Iterative Method is

$$x_{k+1} = \phi(x_k)$$

put $k = 0$

$$x_{0+1} = \phi(x_0) \Rightarrow x_1 = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.5+1}}$$

$$= \frac{1}{\sqrt{1.5}} = \frac{1}{1.22474448} = 0.8164966$$

put $k = 1$

$$x_2 = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.8164966+1}} = \frac{1}{\sqrt{1.8164966}}$$

$$= \frac{1}{1.3477746} = 0.7419638$$

put $k = 2$

$$x_3 = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.7419638+1}} = \frac{1}{\sqrt{1.7419638}} = \frac{1}{1.3198347}$$

for $k=3$

$x_4 = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.7576706+1}} = \frac{1}{\sqrt{1.7576706}}$

$= \frac{1}{1.3257716} = 0.7542777$

So a real root of the equation $x^3+x^2-1=0$ correct up to two decimal places is 0.75

H.W.

(Q) Find a real root of the equation $2x=\cos x+3$ correct to two decimal places using fixed point iteration method. Show that

$\phi(x) = \frac{1}{2}(\cos x+3)$

0.5 H.W.

(Q) Finding the square root of 3 using Fixed Point Iteration Method (correct up to two decimal places)

H.W

(Q) Finding the square root of 2 using Fixed Point Iteration Method (correct up to two decimal places)

H.W.

(Q) Find a real root of the equation $e^x-3x=0$ correct up to three decimal places by Fixed Point Iteration Method starting with $x_0=0.6$

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Method	Iterative formulae	Order of convergence
Bisection	$x_{n+1} = \frac{x_n + x_{n-1}}{2}$ Either $[x_{n-1}, x_{n+1}]$ or $[x_{n+1}, x_n]$ encloses a root	1
False position	$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$	1
Newton-Raphson	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	2
secant	$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \times f(x_n)$	1.618
Fixed Iteration	$x_{i+1} = \phi(x_i)$	1

Numerical solutions of system of Linear equations:

Gauss-Jacobi Method:-

Let the system of equations be

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad (1)$$

Where the diagonal coefficients a_{ii} $(i=1,2,3\cdots n)$ do not vanish. If this is not the case then the rearrange the above system of equations in such a way that the above condition holds. we further assume that the diagonal coefficients a_{ii} are dominant.

We can rewrite the system of equations (1) as

$a_{11}x_1 = b_1 - a_{12}x_2 - a_{13}x_3 \cdots - a_{1n}x_n$

$$\left.\begin{aligned} \Rightarrow x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}x_2}{a_{11}} - \frac{a_{13}x_3}{a_{11}} - \cdots - \frac{a_{1n}x_n}{a_{11}} \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}x_1}{a_{22}} - \frac{a_{23}x_3}{a_{22}} - \cdots - \frac{a_{2n}x_n}{a_{22}} \\ &\cdots\cdots\cdots\cdots \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}x_2}{a_{nn}} - \cdots - \frac{a_{n(n-1)}}{a_{nn}}x_{n-1} \end{aligned}\right\} \quad (2)$$

To solve these equations, we make an Initial guess $x_1^{(0)}, x_2^{(0)} \cdots x_n^{(0)}$ for the unknowns $x_1, x_2 \cdots$

get the first approximation as follows

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}\right]\\
x_2^{(1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}\right]\\
x_3^{(1)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)} - \cdots - a_{3n}x_n^{(0)}\right]\\
&\cdots\cdots\cdots\\
x_n^{(1)} &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} - \cdots - a_{n(n-1)}x_{n-1}^{(0)}\right]
\end{aligned}\right\} \quad (3)$$

Again substituting $x_1^{(1)}, x_2^{(1)}, \ldots x_n^{(1)}$ into the right hand side of (3) we get 2nd approximation and this process is continued until the successive iterations have converged to the required number of significant figures. If $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots x_n^{(k)}$ are the kth approximation then the next approximation is

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

$$\cdots\cdots\cdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots a_{n(n-1)}x_{n-1}^{(k)}\right]$$

$K = 0, 1, 2 \ldots$

This iterative method is called Jacobi iteration and is known as the method of simultaneous displacement. The above method converges

Diagonally Dominant Matrix:

A Square matrix A is said to be diagonally dominant matrix. If $|a_{ii}| \geq \sum_{i \neq j} |a_{ij}| \;\forall\; i \;(i=1,2,3 \ldots n)$

(3) i.e. $|a_{11}| \geq |a_{12}| + |a_{13}| + \cdots + |a_{1n}|$

$|a_{22}| \geq |a_{21}| + |a_{23}| + \cdots + |a_{2n}|$

$\vdots$

$|a_{nn}| \geq |a_{n1}| + |a_{n2}| + \cdots + |a_{n(n-1)}|$

Ex: $A = \begin{bmatrix} 6 & -5 & 1 \\ 4 & -10 & -3 \\ 16 & -3 & 40 \end{bmatrix}$ is diagonally dominant matrix.

Soln: $|a_{11}| \geq |a_{12}| + |a_{13}|$

$= |6| \geq |-5| + |1|$

$= 6 \geq 5+1 \Rightarrow 6 \geq 6$

$|a_{22}| \geq |a_{21}| + |a_{23}|$

$|-10| \geq |4| + |-3|$

$\Rightarrow 10 \geq 4+3 =$

$\Rightarrow 10 \geq 7$

$|a_{33}| \geq |a_{31}| + |a_{32}|$

$|40| \geq |16| + |-3|$

(Q) Starting with $(x^{(0)}, y^{(0)}, z^{(0)}) = (0,0,0)$ and using Jacobi's method apply three iterations for the system

$5x-y+z=10, \quad 2x+8y-z=11, \quad -x+y+4z=3$

Solution:

The given equations can be written as

$5x-y+z=10$

$\Rightarrow 5x = 10+y-z$

$\Rightarrow x = \frac{1}{5}[10+y-z]$

$2x+8y-z=11$

$8y = 11-2x+z$

$\Rightarrow y = \frac{1}{8}[11-2x+z]$

$-x+y+4z=3$

$\Rightarrow 4z = 3+x-y$

$\Rightarrow z = \frac{1}{4}[3+x-y]$

Jacobi's method is applicable here. Under this scheme, we have

$x^{n+1} = \frac{1}{5}[10+y^{(n)}-z^{(n)}]$

$y^{n+1} = \frac{1}{8}[11-2x^{(n)}+z^{(n)}]$

$z^{n+1} = \frac{1}{4}[3+x^{(n)}-y^{(n)}], \quad n=0,1,2\ldots$

Here $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Taking $n=0$ we get

$$x^{(0+1)} = \frac{1}{5}\left[10 + y^{(0)} - z^{(0)}\right]$$

$$x^{(1)} = \frac{1}{5}\left[10 + 0 - 0\right] = \frac{10}{5} = 2$$

$$y^{(0+1)} = \frac{1}{8}\left[11 - 2x^{(0)} + z^{(0)}\right]$$

$$y^{(1)} = \frac{1}{8}\left[11 - 0 + 0\right] = \frac{11}{8} =$$

$$z^{(0+1)} = \frac{1}{4}\left[3 + x^{(0)} - y^{(0)}\right]$$

$$z^{(1)} = \frac{1}{4}\left\{3 + 0 - 0\right] = \frac{3}{4}$$

Next, for $n=1$ we get

$$x^{(1+1)} = \frac{1}{5}\left[10 + y^{(1)} - z^{(1)}\right]$$

$$x^{(2)} = \frac{1}{5}\left[10 + \frac{11}{8} - \frac{3}{4}\right]$$

$$x^{(2)} = \frac{1}{5}\left[\frac{80+11-6}{8}\right] = \frac{1}{5}\left(\frac{85}{8}\right) = \frac{85}{40} = 2.125$$

$$y^{(1+1)} = \frac{1}{8}\left[11 - 2x^{(1)} + z^{(1)}\right]$$

$$y^{(2)} = \frac{1}{8}\left[11 - 2 \times 2 + \frac{3}{4}\right]$$

$$= \frac{1}{8}\left[11 - 4 + \frac{3}{4}\right]$$

$$= \frac{1}{8}\left[\frac{44-16+3}{4}\right]$$

$$= \frac{1}{8} \times \frac{31}{4} = \frac{31}{32} = 0.97$$

$$z^{(1+1)} = \frac{1}{4}\left[3 + x^{(1)} - y^{(1)}\right]$$

$$= \frac{1}{4}\left[\frac{24+16-11}{\ldots}\right]$$

$z^{(2)} = \frac{1}{4} \times \frac{29}{8} = \frac{29}{32} = 0.90$

And for $n=2$

$x^{2+1} = \frac{1}{5}[10 + y^{(2)} - z^{(2)}]$

$x^{(3)} = \frac{1}{5}[10 + 0.97 - 0.90]$

$= \frac{1}{5}[10 + 0.07]$

$x^{(3)} = \frac{10.07}{5} = 2.01$

$y^{2+1} = \frac{1}{8}[11 - 2x^{(2)} + z^{(2)}]$

$= \frac{1}{8}[11 - 2 \times 2.125 + 0.90]$

$y^{(3)} = 0.956$

$z^{2+1} = \frac{1}{4}[3 + x^{(2)} - y^{(2)}]$

$= \frac{1}{4}[3 + 2.125 - 0.97]$

$z^{(3)} = 1.038$

$x^{(3)} = 2.01,\ y^{(3)} = 0.956,\ z^{(3)} = 1.038$ (Ans)

Exact solution is

$x = 2,\ y = 1,\ z = 1$

(Q) Solve the following system of equations by Jacobi iteration method starting with initial guess $x^{(0)}=0,\ y^{(0)}=0,\ z^{(0)}=0$

$$10x+y+z=12$$
$$x+10y+z=12$$
$$x+y+10z=12$$

Solution:

In the given system of equations the coefficient matrix A is

$$A=\begin{bmatrix}10 & 1 & 1\\ 1 & 10 & 1\\ 1 & 1 & 10\end{bmatrix}$$

$|a_{11}| \geq |a_{12}| + |a_{13}|$

$10 \geq 1+1 \Rightarrow 10 \geq 2$

$|a_{22}| \geq |a_{21}| + |a_{23}|$

$10 \geq 1+1 \Rightarrow 10 \geq 2$

$|a_{33}| \geq |a_{31}| + |a_{32}|$

$10 \geq 1+1 \Rightarrow 10 \geq 2$

The given equation can be written as

$$10x+y+z=12$$

$\Rightarrow 10x = 12-y-z$

$\Rightarrow x = \frac{1}{10}[12-y-z]$

$x+10y+z=12$

$x+y+10z=12$

$\Rightarrow 10z=12-x-y$

$\Rightarrow z=\frac{1}{10}[12-x-y]$

Jacobi's method is applicable here under this scheme we have

$x^{n+1}=\frac{1}{10}[12-y^{(n)}-z^{(n)}]$

$y^{n+1}=\frac{1}{10}[12-x^{(n)}-z^{(n)}]$

$z^{n+1}=\frac{1}{10}[12-x^{(n)}-y^{(n)}]$

Here $x^{(0)}=y^{(0)}=z^{(0)}=0$

Taking $n=0$ we get

$x^{0+1}=\frac{1}{10}[12-y^{(0)}-z^{(0)}]$

$x^{(1)}=\frac{1}{10}[12-0-0]$

$=\frac{1}{10}\times 12=\frac{12}{10}=1.2$

$y^{0+1}=\frac{1}{10}[12-x^{(0)}-z^{(0)}]$

$y^{(1)}=\frac{1}{10}[12-0-0]$

$=\frac{1}{10}\times 12=\frac{12}{10}=1.2$

$z^{0+1}=\frac{1}{10}[12-x^{(0)}-y^{(0)}]$

$z^{(1)}=\frac{1}{10}[12-0-0]$

$=\frac{1}{10}\times 12=\frac{12}{10}=1.2$

$\ldots 1.2)$

for $n = 1$, we get

$$x^{(1+1)} = \frac{1}{10}[12 - y^{(1)} - z^{(1)}]$$

$$x^{(2)} = \frac{1}{10}[12 - 1.2 - 1.2]$$

$$= \frac{9.6}{10} = 0.96$$

$$y^{(1+1)} = \frac{1}{10}[12 - x^{(1)} - z^{(1)}]$$

$$y^{(2)} = \frac{1}{10}[12 - 1.2 - 1.2]$$

$$= \frac{9.6}{10} = 0.96$$

$$z^{(1+1)} = \frac{1}{10}[12 - x^{(1)} - y^{(1)}]$$

$$z^{(2)} = \frac{1}{10}[12 - 1.2 - 1.2]$$

$$= \frac{9.6}{10} = 0.96$$

$$(x^{(2)}, y^{(2)}, z^{(2)}) = (0.96, 0.96, 0.96)$$

Similarly $(x^{(3)}, y^{(3)}, z^{(3)}) = (1.008, 1.008, 1.008)$

$$(x^{(4)}, y^{(4)}, z^{(4)}) = (0.9984, 0.9984, 0.9984)$$

$$(x^{(5)}, y^{(5)}, z^{(5)}) = (1.0003, 1.0003, 1.0003)$$

$$(x^{(6)}, y^{(6)}, z^{(6)}) = (0.9999, 0.9999, 0.9999)$$

$$(x^{(7)}, y^{(7)}, z^{(7)}) = (1.0000, 1.0000, 1.0000)$$

Exact solution is

$x = 1$, $y = 1$, $z = 1$.

H.W

(Q) Solve the following system of linear equa[tions] using Gauss-Jacobi method up to four iterations $2x+y+z=5$, $3x+5y+2z=15$, $2x+y+4z=8$

H.W

(Q) Solve the following system of linear equations using Gauss-Jacobi method up to seven Iterations

$$8x_1+2x_2-2x_3=8$$

$$x_1-8x_2+3x_3=-4$$

$$2x_1+x_2+9x_3=12$$

H.W

(Q) Solve the following linear system by Jacobi method with Initial guess $x^{(0)}=y^{(0)}=z^{(0)}=0$ find out three Iterations

$$10x+3y+z=14$$

$$2x_1 \cancel{-10x_2+3x_3=-5}$$

$$2x-10y+3z=-5$$

$$x+3y+10z=14$$

Gauss-Seidel Method:-

The general form of Gauss-Seidel iteration method is

$$x_1^{(K+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(K)} - a_{13}x_3^{(K)} - \cdots - a_{1n}x_n^{(K)}\right]$$

$$x_2^{(K+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(K+1)} - a_{23}x_3^{(K)} - \cdots - a_{2n}x_n^{(K)}\right]$$

$$x_3^{(K+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(K+1)} - a_{32}x_2^{(K+1)} - \cdots - a_{3n}x_n^{(K)}\right]$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$x_n^{(K+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(K+1)} - a_{n2}x_2^{(K+1)} - \cdots - a_{nn-1}x_{n-1}^{(K+1)}\right]$$

Gauss-Seidel method is also called the method of sucessive displacements. Gauss-Seidel method converges twice as fast as Jacobi's method.

(Q) Solve the following system of linear equations by Gauss-Seidel method starting with initial guess $x_1^{(0)}=0,\ x_2^{(0)}=0,\ x_3^{(0)}=0$

$$10x_1 + x_2 + x_3 = 12$$

$$x_1 + 10x_2 + x_3 = 12$$

$$x_1 + x_2 + 10x_3 = 12$$

Solution:-

Given that the system of linear equation can be written as

$$10x_1 + x_2 + x_3 = 12$$

$x_1 + 10x_2 + x_3 = 12$

$\Rightarrow 10x_2 = 12 - x_1 - x_3$

$\Rightarrow x_2 = \frac{1}{10}[12 - x_1 - x_3]$

$x_1 + x_2 + 10x_3 = 12$

$\Rightarrow 10x_3 = 12 - x_1 - x_2$

$\Rightarrow x_3 = \frac{1}{10}[12 - x_1 - x_2]$

Gauss-Seidel method is applicable here. Under this scheme we have

$x_1^{(n+1)} = \frac{1}{10}[12 - x_2^{(n)} - x_3^{(n)}]$

$x_2^{(n+1)} = \frac{1}{10}[12 - x_1^{(n+1)} - x_3^{(n)}]$

$x_3^{(n+1)} = \frac{1}{10}[12 - x_1^{(n+1)} - x_2^{(n+1)}]$

$n = 0, 1, 2 \cdots$

Here $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

Taking $n=0$ we get

$x_1^{(0+1)} = \frac{1}{10}[12 - x_2^{(0)} - x_3^{(0)}]$

$x^{(1)} = \frac{1}{10}[12 - 0 - 0]$

$= \frac{12}{10} = 1.2$

$x_2^{(0+1)} = \frac{1}{10}[12 - x_1^{(0+1)} - x_3^{(0)}]$

$x_2^{(1)} = \frac{1}{10}[12 - x_1^{(1)} - x_3^{(0)}]$

$= \frac{1}{10}[12 - 1.2 - 0]$

$= \cdot \quad 1.08$

$$x_3^{(0+1)} = \frac{1}{10}\left[12 - x_1^{(0+1)} - x_2^{(0+1)}\right]$$

$$x_3^{(1)} = \frac{1}{10}\left[12 - x_1^{(1)} - x_2^{(1)}\right]$$

$$= \frac{1}{10}\left[12 - 1.2 - 1.08\right] = 0.972$$

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (1.2, 1.08, 0.972)$$

Next for $n = 1$

$$x_1^{(1+1)} = \frac{1}{10}\left[12 - x_2^{(1)} - x_3^{(1)}\right]$$

$$x_1^{(2)} = \frac{1}{10}\left[12 - 1.08 - 0.972\right] = 0.995$$

$$x_2^{(1+1)} = \frac{1}{10}\left[12 - x_1^{(1+1)} - x_3^{(1)}\right]$$

$$x_2^{(2)} = \frac{1}{10}\left[12 - x_1^{(2)} - x_3^{(1)}\right]$$

$$= \frac{1}{10}\left[12 - 0.995 - 0.972\right] = 1.003$$

$$x_2^{(1+1)} = \frac{1}{10}\left[12 - x_1^{(1+1)} - x_2^{(1+1)}\right]$$

$$x_3^{(2)} = \frac{1}{10}\left[12 - x_1^{(2)} - x_2^{(2)}\right]$$

$$= \frac{1}{10}\left[12 - 0.995 - 1.003\right] = 1.000$$

Exact solution is $x_1 = 1, x_2 = 1, x_3 = 1$.

(Q) Starting with $(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$ and using Gauss-Seidel method apply two iterations for the system

$$9x + 4y + z = -17$$

$$x + 6y = 4$$

Solution:

Given that the system of linear equations can be written as

$$9x+4y+z=-17$$

$$\Rightarrow 9x=-17-4y-z$$

$$\Rightarrow x=\frac{1}{9}[-17-4y-z]$$

$$x+6y=4$$

$$\Rightarrow 6y=4-x$$

$$\Rightarrow y=\frac{1}{6}(4-x)$$

$$x-2y-6z=14$$

$$\Rightarrow -6z=14-x+2y$$

$$\Rightarrow 6z=-14+x-2y$$

$$\Rightarrow z=\frac{1}{6}[x-2y-14]$$

Gauss-Seidel method is applicable here

Under this scheme we have

$$x^{(n+1)}=\frac{1}{9}[-17-4y^{(n)}-z^{(n)}]$$

$$y^{(n+1)}=\frac{1}{6}[4-x^{(n+1)}]$$

$$z^{(n+1)}=\frac{1}{6}[x^{(n+1)}-2y^{(n+1)}-14]$$

$n=0,1,2\ldots$

Here $x^{(0)}=y^{(0)}=z^{(0)}=0$

Taking $n=0$ we get

$$x^{0+1}=\frac{1}{9}[17-4y^{(0)}-z^{(0)}]$$

$$\ldots=\frac{-17}{9}=-1.889$$

$$y^{(0+1)} = \frac{1}{6}\left[4 - x^{(0+1)}\right]$$

$$y^{(1)} = \frac{1}{6}\left[4 - x^{(1)}\right]$$

$$= \frac{1}{6}\left[4 - (-1.889)\right]$$

$$= \frac{1}{6}\left[4 + 1.889\right] = 0.9815$$

$$z^{(0+1)} = \frac{1}{6}\left[x^{(0+1)} - 2y^{(0+1)} - 14\right]$$

$$z^{(1)} = \frac{1}{6}\left[x^{(1)} - 2y^{(1)} - 14\right]$$

$$= \frac{1}{6}\left[-1.889 - 2\times(0.9815) - 14\right]$$

$$= -2.975$$

for $n=1$

$$x^{(1+1)} = \frac{1}{9}\left[-17 - 4y^{(1)} - z^{(1)}\right]$$

$$x^{(2)} = \frac{1}{9}\left[-17 - 4\times 0.9815 - (-2.975)\right]$$

$$= -1.995$$

$$y^{(1+1)} = \frac{1}{6}\left[4 - x^{(1+1)}\right]$$

$$= \frac{1}{6}\left[4 - x^{(2)}\right]$$

$$y^{(2)} = \frac{1}{6}\left[4 - (-1.995)\right] = 0.9991$$

$$z^{(1+1)} = \frac{1}{6}\left[x^{(1+1)} - 2y^{(1+1)} - 14\right]$$

$$z^{(2)} = \frac{1}{6}\left[x^{(2)} - 2y^{(2)} - 14\right]$$

$$= \frac{1}{6}\left[-1.995 - 2\times(0.9991) - 14\right]$$

H.W

(Q) Starting with $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0,0,0)$ solve the following system of linear equations Gauss-Seidel method

$$8x_1 + 2x_2 - 2x_3 = 8$$
$$x_1 - 8x_2 - 3x_3 = -4$$
$$2x_1 + x_2 + 9x_3 = 12$$

H.W

(Q) Solve the following system of linear equations by Gauss-Seidel method apply seven Iterations for the system

$$2x_1 - x_2 + 0.x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0.x_1 - x_2 + 2x_3 = 1$$

H.W

(Q) Solve the following system of linear equations by Gauss-Seidel method apply three Iterations for the system

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Doolittle's method:-

Doolittle's method LU factorization of A when the diagonal elements of lower triangular matrix, L have a unit value.

Step:-

(1) create matrices A, X and B, where A is the augmented matrix; X constitutes the variable vectors and B are the constants.

(2) Let A = LU where L is the Lower triangular matrix (LTM) and U is the upper triangular matrix (UTM) assume that the diagonal entries L is equal to 1.

(3) Let LY = B, solve the Y's

(4) Let UX = Y solve for the variable vectors X.

(Q) Solve by Doolittle's method the system of equation

$$x_1 + x_2 + x_3 = 5$$
$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 8$$

Solution: Step-1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$$

Step-2

Let $A = LU$

$$\begin{bmatrix} 1 & 1 & 1 \\ \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix}$$

$d=1, \quad e=1, \quad f=1$

$ad=1 \qquad ae+g=2 \quad , \quad af+h=2$

$a\times(1)=1 \qquad (1)\times(1)+g=2 \qquad (1)\times(1)+h=2$

$\boxed{a=1} \qquad 1+g=2 \qquad 1+h=2$

$\Rightarrow \boxed{g=2-1=1} \qquad \boxed{h=2-1=1}$

$bd=1 \qquad be+cg=2 \qquad bf+ch+i=3$

$b\times(1)=1 \qquad (1)\times(1)+c(1)=2 \qquad (1)\times(1)+(1)\times(1)+i=3$

$\boxed{b=1} \qquad 1+c=2 \qquad \Rightarrow 1+1+i=3$

$\Rightarrow \boxed{c=2-1=1} \qquad \Rightarrow 2+i=3$

$\Rightarrow \boxed{i=3-2=1}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step-3

Let $LY=B$ $\qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$$

$y_1=5$

$y_1+y_2=6; \qquad y_1+y_2+y_3=8$

$\Rightarrow 5+1+y_3=8$

Step-4

Let $UX = Y$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$x_3 = 2$, $x_2 + x_3 = 1$

$\Rightarrow x_2 + 2 = 1$

$\Rightarrow x_2 = 1 - 2 = -1$

$x_1 + x_2 + x_3 = 5$

$\Rightarrow x_1 + (-1) + 2 = 5$

$\Rightarrow x_1 + 1 = 5 \Rightarrow x_1 = 5 - 1 = 4$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

(Q) Solve by Doolittle's method the system of equation

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

Solution: step-1

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Step:- 2

let $A = LU$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix}$$

$d = 2$, $e = 3$, $f = 1$

$ad = 1$ $\quad ae+g = 2$, $\quad af+h = 3$

$a \times (2) = 1$ $\quad \frac{1}{2} \times (3) + g = 2$ $\quad \Rightarrow \frac{1}{2} \times 1 + h = 3$

$2a = 1$ $\quad \frac{3}{2} + g = 2$ $\quad \Rightarrow 1/2 + h = 3$

$\Rightarrow \boxed{a = \frac{1}{2}}$ $\quad \boxed{g = 2 - \frac{3}{2} = \frac{4-3}{2} = \frac{1}{2}}$ $\quad \Rightarrow h = 3 - \frac{1}{2}$

$\Rightarrow \boxed{h = \frac{6-1}{2} = \frac{5}{2}}$

$bd = 3$ $\quad be + cg = 1$

$b \times (2) = 3$ $\quad \frac{3}{2} \times 3 + (\frac{1}{2}) \times c = 1$

$2b = 3$ $\quad \Rightarrow \frac{9}{2} + \frac{c}{2} = 1$

$\Rightarrow \boxed{b = 3/2}$ $\quad \Rightarrow \frac{9+c}{2} = 1$

$\Rightarrow 9 + c = 2$

$\Rightarrow \boxed{c = 2 - 9 = -7}$

$bf + ch + i = 2$

$\Rightarrow \frac{3}{2} \times 1 + (-7) \times \frac{5}{2} + i = 2$

$2x^2 = 4$

$2 + 2\dot{2} = 2X_2 = 4$

$4 + 32 = 36$

36

$\boxed{\frac{36}{2} = 18}$

$LY = B$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$y_1 = 9$

$\frac{1}{2}y_1 + y_2 = 6$

$\frac{9}{2} + y_2 = 6$

$y_2 = 6 - \frac{9}{2} = \frac{12-9}{2} = \frac{3}{2}$

$0y_1 + (-7)y_2 + y_3 = 8$

$-7 \times \frac{3}{2} + y_3 = 8$

$-\frac{21}{2} + y_3 = 8$

$\frac{-21 + 2y_3}{2} = 8$

$-21 + 2y_3 = 16$

Step-4

Let $UX = Y$ $\qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$\Rightarrow 18x_3 = 5$

$\Rightarrow x_3 = \frac{5}{18}$

$\frac{1}{2}x_2 + \frac{5}{2}x_3 = \frac{3}{2}$

$\frac{1}{2}x_2 + \frac{5}{2} \times \frac{5}{18} = \frac{3}{2}$

$\frac{1}{2}x_2 + \frac{25}{36} = \frac{3}{2}$

$\Rightarrow \frac{18x_2 + 25}{36} = \frac{3}{2}$

$\Rightarrow 18x_2 + 25 = \frac{3 \times 36}{2} = 54$

$\Rightarrow 18x_2 = 54 - 25$

$\Rightarrow 18x_2 = 29$

$\Rightarrow x_2 = \frac{29}{18}$

$2x_1 + 3x_2 + x_3 = 9$

$2x_1 + 3 \times \frac{29}{18} + \frac{5}{18} = 9$

$\Rightarrow \frac{36x_1 + 87 + 5}{18} = 9$

$\Rightarrow 36x_1 + 92 = 9 \times 18 = 162$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{35}{18} \\ \frac{29}{18} \\ \frac{5}{18} \end{bmatrix}$$

H.W

(Q) solve by LU Decomposition method.

$$3x_1 + 4x_2 + 2x_3 = 15$$
$$5x_1 + 2x_2 + x_3 = 18$$
$$2x_1 + 3x_2 + 2x_3 = 10$$

H.W

(Q) Solve the following system of equation by LU Decomposition method

$$x_1 + 5x_2 + x_3 = 14$$
$$2x_1 + x_2 + 3x_3 = 13$$
$$3x_1 + x_2 + 4x_3 = 17$$

H.W

(Q) Solve the following system by Doolittle's method

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

Crout's method:

Crout's method is LU factorization of A when the diagonal elements of UTM, U have a unit value.

Step:-

1. Create matrices A, X and B, where A is the augmented matrix. X constitutes the variable vectors and B are the constants.

2. Let $A = LU$ where L is the lower triangular matrix (LTM) and U is the upper triangular matrix (UTM) assume that the diagonal entries U is equal to 1.

3. Let $LY = B$ solve for Y's

4. Let $UX = Y$ solve for the variable vectors X.

(Q) Solve the following system by Crout's method.

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + 2x_2 + 5x_3 = 22$$

Solution: Step:-1

The given system of equations can be written in matrix form $AX = B$ as follows

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 14 \\ 18 \\ 22 \end{bmatrix}$$

Step :- 2

Let $A = LU$ such that

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a & ag & ah \\ b & bg+c & bh+ci \\ d & dg+e & dh+ei+f \end{bmatrix}$$

$a = 1$, $ag = 2$
$\Rightarrow 1 \times g = 2$
$\Rightarrow \boxed{g = 2}$

$ah = 3$
$\Rightarrow 1 \times h = 3$
$\Rightarrow \boxed{h = 3}$

$b = 2$, $bg + c = 5$
$2 \times 2 + c = 5$
$4 + c = 5$
$\Rightarrow \boxed{c = 5 - 4 = 1}$

$bh + ci = 2$
$2 \times 3 + 1 \times i = 2$
$\Rightarrow 6 + i = 2$
$\Rightarrow i = 2 - 6 = -4$

$d = 3$, $dg + e = 2$
$\Rightarrow 3 \times 2 + e = 2$

$dh + ei + f = 5$

$\Rightarrow 3 \times 3 + (-4) \times (-4) + f = 5$

$\Rightarrow 9 + 16 + f = 5$

$\Rightarrow 25 + f = 5$

$\Rightarrow \boxed{f = 5 - 25 = -20}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & -20 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Step : 3

let $LY = B$ $\quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & -20 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 22 \end{bmatrix}$$

$y_1 = 14$

$2y_1 + y_2 = 18$

$2 \times 14 + y_2 = 18$

$\Rightarrow 28 + y_2 = 18$

$\Rightarrow y_2 = 18 - 28 = -10$

$3y_1 - 4y_2 - 20y_3 = 22$

$3 \times 14 - 4(-10) - 20y_3 = 22$

$\Rightarrow 42 + 40 - 20y_3 = 22$

$\Rightarrow 82 - 20y_3 = 22$

$\Rightarrow -20y_3 = 22 - 82 = -60$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ 3 \end{bmatrix}$$

Step:-4

Let $UX = Y$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ 3 \end{bmatrix}$$

$x_3 = 3$

$x_2 - 4x_3 = -10$

$\Rightarrow x_2 - 4 \times 3 = -10$

$\Rightarrow x_2 - 12 = -10$

$\Rightarrow x_2 = -10 + 12 = 2$

$x_1 + 2x_2 + 3x_3 = 14$

$\Rightarrow x_1 + 2 \times (2) + 3 \times (3) = 14$

$\Rightarrow x_1 + 4 + 9 = 14$

$\Rightarrow x_1 + 13 = 14$

$\Rightarrow x_1 = 14 - 13 = 1$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(Q) Solve by Crout's method the system of equations

$$2x_1 + 3x_2 + x_3 = -1$$

$$5x_1 + x_2 + x_3 = 9$$

$$3x_1 + 2x_2 + 4x_3 = 11$$

Solution: Step-1

let $AX = B$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

Step-2

let $A = LU$ such that

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} a & ag & ah \\ b & bg+c & bh+ci \\ d & dg+e & dh+ei+f \end{bmatrix}$$

$a = 2$

$ag = 3$
$\Rightarrow 2 \times g = 3$
$\Rightarrow 2g = 3$
$\Rightarrow \boxed{g = \frac{3}{2}}$

$ah = 1$
$2 \times h = 1$
$\Rightarrow 2h = 1$
$\Rightarrow \boxed{h = \frac{1}{2}}$

$b=5$ $\quad bg+c=1$

$5\times\frac{3}{2}+c=1$

$\Rightarrow \frac{15}{2}+c=1$

$\Rightarrow c=1-\frac{15}{2}=\frac{2-15}{2}=\frac{-13}{2}$

$\Rightarrow \boxed{c=\frac{-13}{2}}$

$bh+ci=1$

$5\times\frac{1}{2}+\left(\frac{-13}{2}\right)i=1$

$\Rightarrow \frac{5}{2}-\frac{13}{2}i=1$

$\Rightarrow \frac{5-13i}{2}=1$

$\Rightarrow 5-13i=2$

$\Rightarrow -13i=2-5$

$\Rightarrow -13i=-3$

$\Rightarrow \boxed{i=\frac{3}{13}}$

$d=3$ $\quad dg+e=2$

$\Rightarrow 3\times\frac{3}{2}+e=2$

$\Rightarrow \frac{9}{2}+e=2$

$\Rightarrow e=2-\frac{9}{2}=\frac{4-9}{2}=\frac{-5}{2}$

$\Rightarrow \boxed{e=-\frac{5}{2}}$

$dh+ei+f=4$

$3\times\frac{1}{2}+\left(\frac{-5}{2}\right)\times\frac{3}{13}+f=4$

$\Rightarrow \frac{3}{2}-\frac{15}{26}+f=4$

$\Rightarrow \frac{39-15+26f}{26}=4$

$\Rightarrow 39-15+26f=4\times26$

$\Rightarrow 39-15+26f=104$

$\Rightarrow 24+26f=104$

$\Rightarrow 26f=104-24=80$

$\Rightarrow \boxed{f=\frac{80}{26}=\frac{40}{13}}$

$l=\begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$ $\quad U=\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \end{bmatrix}$

Step : 3

let $LY = B$ $\qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & -\frac{13}{2} & 0 \\ 3 & -\frac{5}{2} & \frac{40}{13} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$\Rightarrow 2y_1 = -1$

$\Rightarrow y_1 = -\frac{1}{2}$

$5y_1 - \frac{13}{2} y_2 = 9$

$\Rightarrow 5 \times (-\frac{1}{2}) - \frac{13}{2} y_2 = 9$

$\Rightarrow -\frac{5}{2} - \frac{13}{2} y_2 = 9$

$\Rightarrow \frac{-5 - 13y_2}{2} = 9$

$\Rightarrow -5 - 13y_2 = 18$

$\Rightarrow -13y_2 = 18 + 5 = 23$

$\Rightarrow y_2 = -\frac{23}{13}$

$3y_1 - \frac{5}{2} y_2 + \frac{40}{13} y_3 = 11$

$3 \times (-\frac{1}{2}) - \frac{5}{2} \times (-\frac{23}{13}) + \frac{40}{13} y_3 = 11$

$-\frac{3}{2} + \frac{115}{26} + \frac{40}{13} y_3 = 11$

$\Rightarrow \frac{-39 + 115 + 80y_3}{26} = 11$

$\Rightarrow 76 + 80y_3 = 11 \times 26 = 286$

$\Rightarrow 80y_3 = 286 - 76 = \cancel{132}\ 210$

$\Rightarrow y_3 = \frac{210}{80} = \cancel{\frac{333}{200}}\ \frac{21}{8}$

Step 4

Let $UX = y$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{13} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{-23}{13} \\ \frac{21}{8} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_3 = \frac{21}{8}$ $\Rightarrow x_2 + \frac{3}{13} x_3 = -\frac{23}{13}$

$x_3 = 2.625$ $\Rightarrow x_2 + \frac{3}{13} \times \frac{21}{8} = -\frac{23}{13}$

$\Rightarrow x_2 + \frac{63}{104} \cancel{\frac{99}{260}} = -\frac{23}{13}$

$\Rightarrow \frac{104 x_2 + 63}{104} = -\frac{23}{13}$

$\Rightarrow \cancel{260 x_2 + 99 = \frac{-23 \times 260}{13}}$

$\Rightarrow \cancel{260 x_2 + 99 = -460}$

$\Rightarrow \cancel{260 x_2 = -460 - 99}$ $\Rightarrow 104 x_2 + 63 = -\frac{23 \times 104}{13}$

$\Rightarrow \cancel{260 x_2 = -559}$ $\Rightarrow 104 x_2 + 63 = -184$

$\Rightarrow \cancel{x_2 = \frac{-559}{260}} = -2.15$ $\Rightarrow 104 x_2 = -184 - 63$

$= -247$

$x_1 + \frac{3}{2} x_2 + \frac{1}{2} x_3 = -\frac{1}{2}$ $\Rightarrow x_2 = \frac{-247}{104} = -2.375$

$x_1 + \frac{3}{2} \times (-2.375) + \frac{1}{2} \times (2.625) = -\frac{1}{2}$

$\Rightarrow x_1 - \frac{7.125}{2} + \frac{2.625}{2} = -\frac{1}{2}$

$\Rightarrow \frac{2x_1 - 7.125 + 2.625}{2} = -\frac{1}{2}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.75 \\ -2.375 \\ 2.625 \end{bmatrix}$$

H.W

(Q) solve by crout's method

$$3x_1 + x_2 + x_3 = 4$$
$$x_1 + 2x_2 + 2x_3 = 3$$
$$2x_1 + x_2 + 3x_3 = 4$$

H.W

(Q) solve by crout's method

$$x_1 + 3x_2 + 8x_3 = 4$$
$$x_1 + 4x_2 + 3x_3 = -2$$
$$x_1 + 3x_2 + 4x_3 = 1$$

H.W

(Q) solve by crout's method

$$3x_1 - x_2 + 2x_3 = 12$$
$$x_1 + 2x_2 + 3x_3 = 11$$
$$2x_1 - 2x_2 - x_3 = 2$$

H.W

(Q) solve by crout's method

$$10x_1 + 2x_2 + x_3 = 9$$
$$x_1 + 10x_2 - x_3 = -22$$
$$-2x_1 + 3x_2 + 10x_3 = 22$$

Cholesky's Method:-

The cholesky's method unlike the Doolittle and crout's does not have any condition for the main diagonal entries. The matrix should be symmetric and for a symmetric positive definite matrix.

Step:-

(1) create matrix A, X and B.

(2) Let $A = LL^T$; $U = L^T$

(3) Let $LY = B$ solve for Y's

(4) $L^T X = Y$ then solve for X.

(Q) Solve by cholesky's method

$$4x_1 + 10x_2 + 8x_3 = 44$$
$$10x_1 + 26x_2 + 26x_3 = 128$$
$$8x_1 + 26x_2 + 61x_3 = 214$$

Solution: Step-1

Let $AX = B$

$$A = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Step-2

Let $A = LL^T$ such that

$$A = \begin{bmatrix} 4 & 10 & 8 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & d \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$$

$a^2 = 4$
$\Rightarrow \boxed{a = \sqrt{4} = 2}$

$ab = 10$
$\Rightarrow 2b = 10$
$\Rightarrow \boxed{b = \frac{10}{2} = 5}$

$ad = 8$
$\Rightarrow 2d = 8$
$\Rightarrow \boxed{d = \frac{8}{2} = 4}$

$b^2 + c^2 = 26$
$\Rightarrow (5)^2 + c^2 = 26$
$\Rightarrow 25 + c^2 = 26$
$\Rightarrow c^2 = 26 - 25 = 1$
$\Rightarrow \boxed{c = 1}$

$bd + ce = 26$
$\Rightarrow 5 \times 4 + 1 \times e = 26$
$\Rightarrow 20 + e = 26$
$\Rightarrow \boxed{e = 26 - 20 = 6}$

$d^2 + e^2 + f^2 = 61$
$(4)^2 + (6)^2 + f^2 = 61$
$\Rightarrow 16 + 36 + f^2 = 61$
$\Rightarrow 52 + f^2 = 61$
$\Rightarrow f^2 = 61 - 52 = 9$
$\Rightarrow \boxed{f = \sqrt{9} = 3}$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \qquad L^T = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Step : 3
Let $LY = B$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$