

# **FINITE ELEMENT METHOD ( FEM )**

SEMESTER : 5TH  
BRANCH : MECHANICAL ENGG.

## **MODULE - I**

*Short-Type Questions & Answers  
Long-Type Questions & Answers*

**BIJAN KUMAR GIRI**

# **SHORT-TYPE**

# **Questions & Answers**

*Bijan Kumar Giri*

## **Q. What is Finite Element Method .**

Ans : The **finite element method (FEM)** is the most widely used method for solving problems of engineering and mathematical models. Typical problem areas of interest include the traditional fields of structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

The FEM is a particular numerical method for obtaining approximate solution of many problems encountered in engineering analysis using two or three space variables (i.e., some boundary value problems).

## **Q. What is the basic approach of solving problems using Finite Element Method(FEM) ?**

Ans : To solve a problem, the FEM subdivides a large system into smaller, simpler parts that are called **finite elements**. This is achieved by a particular space discretization in the space dimensions, which is implemented by the construction of a mesh of the object: the numerical domain for the solution, which has a finite number of points. The finite element method formulation of a boundary value problem finally results in a system of algebraic equations . The solution of these equations gives us the approximate behaviour of the continuum .

## **Q. Mention few applications of FEM .**

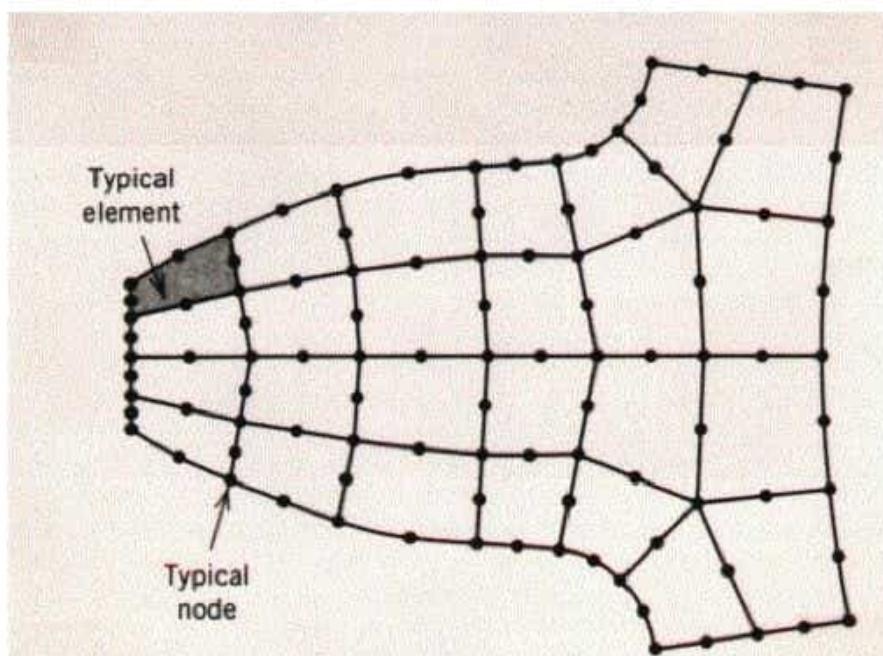
Ans : The applications of FEM ranges from deformation and engineering stress analysis of automotives , aircrafts, buildings ,bridges , structures , trusses to field analysis of heat flow/transfer , magnetic and electric field , fluid flow , seepage , duct , and also axisymmetric problems .

**Q. What are the steps of FEM for analysis a complex structure**

### ***FEM in Structural Analysis (The Procedure)***

- Divide structure into pieces (elements with nodes)
- Describe the behavior of the physical quantities on each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g., displacements)
- Calculate desired quantities (e.g., strains and stresses) at selected elements

***Example:***



FEM model for a gear tooth (From Cook's book, p.2).

## Q. What are the applications of FEM?

- structural mechanics (e.g. building bridges, wind turbines, etc.)
- electromagnetism (yes, Maxwell equations. You can get microwave or light propagation in waveguides, antenna design, etc.)
- heat transfer (e.g. PCB or silicon chip design)
- electrical parasitic extraction (e.g. capacitive touch screens, coils design)
- fluidics (e.g. pipe systems, nano-technology, etc.)
- acoustics (e.g. loudspeaker design, microphone design, etc.)

## Q. What do you mean by FEA in FEM.

The finite element method (FEM) is a numerical technique used to perform finite element analysis (FEA) of any given physical phenomenon.

It is necessary to use mathematics to comprehensively understand and quantify any physical phenomena, such as structural or fluid behavior, thermal transport, wave propagation, and the growth of biological cells. Most of these processes are described using partial differential equations (PDEs). However, for a computer to solve these PDEs, numerical techniques have been developed over the last few decades and one of the most prominent today is the finite element method.

## Q. Define Potential Energy in the context of Finite Element analysis.

Potential Energy: Work done against resistance of deformation

The total potential energy,  $\Pi$  of an elastically body, is defined as the sum of total strain energy ( $U$ ) and the work potential (W.P.).

i.e.  $\Pi = \text{Strain energy} + \text{Work potential}$

$$\Rightarrow \Pi = U + W.P.$$

where  $U = \text{Strain energy}^{\text{Elastic}} = \frac{1}{2} K s^2$

$$W.P. = \text{Work Potential} = F.q$$

Force acting at the node.  $\uparrow$  Nodal displacement

Q. What do you mean by 'Traction force' and 'Body force'?

Ans: Traction force( $T$ ): The distributed force per unit area (for 2-Dimensional) or the distributed force per unit length (for one dimensional) is called traction force or simply traction ( $T$ ).

$$T = [ T_x \quad T_y \quad T_z ]^T$$

where  $T_x$ ,  $T_y$  and  $T_z$  are the components of  $T$ .

Examples: Distributed contact force, Action of pressure

Body force( $f$ ): The distributed force per unit volume is called as body force.

$$f = [ f_x \quad f_y \quad f_z ]$$

where the  $f_x$ ,  $f_y$  and  $f_z$  are the components of ' $f$ '.

Examples: Weight per volume, Gravitational force, Viscous force,

**Q. Write the equilibrium equations in 3-D stress analysis.**

**ANS :** *Equilibrium Equations:*

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x &= 0 , \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y &= 0 , \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z &= 0 ,\end{aligned}\quad (6)$$

or

$$\sigma_{ij,j} + f_i = 0$$

**Q. What do you mean by Plain stress and Plain strain ?**

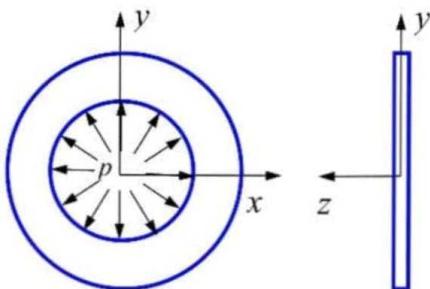
**ANS :**

*Plane (2-D) Problems*

- *Plane stress:*

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \quad (\varepsilon_z \neq 0) \quad (1)$$

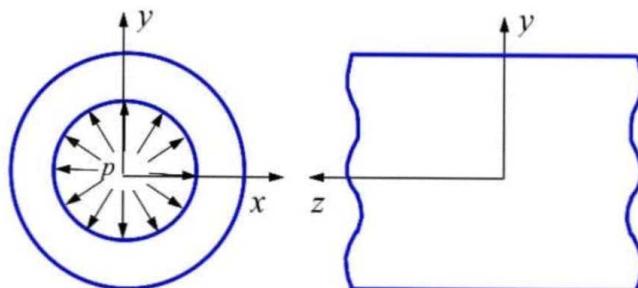
A thin planar structure with constant thickness and loading within the plane of the structure (xy-plane).



- *Plane strain:*

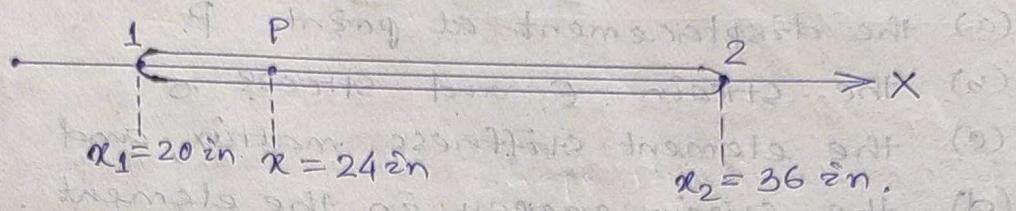
$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \quad (\sigma_z \neq 0) \quad (2)$$

A long structure with a uniform cross section and transverse loading along its length (z-direction).



Problem-1: For the following linear element,

- (a) Evaluate  $\xi$ ,  $N_1$  and  $N_2$  at point P.
- (b) If  $q_1 = 0.003$  in. and  $q_2 = -0.005$  in., determine the value of the displacement ' $u$ ' at point P.



Solution

$$(a) \xi_P = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

$$= \frac{2(24 - 20)}{36 - 20} - 1 = \frac{28}{16} - 1 = 0.5$$

$$N_1 = \frac{1 - \xi_P}{2} = \frac{1 - (-0.5)}{2} = 0.75$$

$$N_2 = \frac{1 + \xi_P}{2} = \frac{1 + (-0.5)}{2} = 0.25$$

- (b) Nodal displacement of point P,  $u_P = N_1 q_1 + N_2 q_2$

$$u_P = (0.75 \times 0.003) + (0.25 \times -0.005)$$

$$= 0.001 \text{ in.}$$

Problem-2: If an element having first node at  $x_1 = 20$  in. & second node at  $x_2 = 36$  in. find  $\xi$ ,  $N_1$ ,  $N_2$  at a point P ( $x = 30$ ) .

$$\xi_P = \frac{2(x - x_1)}{x_2 - x_1} - 1 = \frac{2(30 - 20)}{36 - 20} - 1 = 0.25$$

$$N_1 = \frac{1 - \xi_P}{2} = \frac{1 - 0.25}{2} = 0.375$$

$$N_2 = \frac{1 + \xi_P}{2} = \frac{1 + 0.25}{2} = 0.625$$

$$[281.0 \quad 281.0] = 281$$

Ans

# **LONG -TYPE**

## **Questions & Answers**

*Bijan Kumar Giri*

## **Q. What are the steps of finite elements?**

**Ans :** The major steps in the Finite Element Method,

1. Discretization of real continuum or structure – (Establish the FE mesh)
  - a. Establish the FE mesh with set coordinates, element numbers and node numbers
  - b. The discretized FE model must be situated with a coordinate system
  - c. Elements and nodes in the discretized FE model need to be identified by "element numbers" and "nodal numbers."
  - d. Nodes are identified by the assigned node numbers and their corresponding coordinates
2. Identify primary unknown quantity
  - a. Primary unknown quantity - The first and principal unknown quantity to be obtained by the FEM
  - b. Eg: Stress analysis: Displacement  $\{u\}$  at nodes
  - c. In stress analysis, The primary unknowns are nodal displacements, but secondary unknown quantities include: strains in elements can be obtained by the "strain-displacement relations," and the unknown stresses in the elements by the stress-strain relations (the Hooke's law).
3. Interpolation functions and the derivation of Interpolation functions
  - a. Interpolation function is called "shape function in some literatures
  - b. There are different forms of interpolation functions used in FEM. The elements using the linear interpolation functions are called "Simplex elements" are the simplest form and the most commonly used in FE formulation.
4. Derivation of Element equation
  - a. The element equation relates the induced primary unknown quantity in the analysis with the action.  
Eg. In a structural stress analysis, Force  $\{F\}$  is the action, Displacement  $\{u\}$  at nodes is the primary unknown and Stresses  $\{\sigma\}$  & Strains  $\{\epsilon\}$  are secondary unknown.
  - b. These are the generally two methods used to derive the element equations:
    - i. The Rayleigh-Ritz method, and
    - ii. The Galerkin method

## 5. Derive overall Stiffness Equation

- a. This step assembles all individual element equations derived in Step 4 to provide the "Stiffness equations" for the entire medium.
- b. Mathematically, this equation has the form,  $[K]\{q\} = \{R\}$  where  $[K]$  is overall stiffness matrix.

## 6. Solve for primary unknowns

- a. Use the inverse matrix method to solve the primary unknown quantities  $\{q\}$  at all the nodes from the overall stiffness equations.  $\{q\} = [K]^{-1}\{R\}$
- b. Else, use the Gaussian elimination method or its derivatives to solve nodal quantities  $\{q\}$  from the equation:  $[K]\{q\} = \{R\}$

## 7. Solve for secondary unknowns.

## 8. Display and Interpretation of Results

- a. Tabulation of results
- b. Graphic displays: (1) Static with contours. (2) Animations

## **Q. What are the different types of elements used in FEM ?**

**Ans :** Elements are divided into 3 categories

1D

2D

3D

---

### **1D**

#### **1 TRUSS elements :**

- Two nodes are sufficient to define
- Each node has only translation degrees of freedom no rotational DOF

#### **2 Beam elements :**

- Three nodes required to define
- Each node has 6 DOF

#### **3 BAR elements :**

- Two nodes required to define
- Each node has only 1DOF( translational )

#### **Special elements:**

#### **4 Rigid elements**

- Two node sets , one set will have dependent and one with independent

#### **5 Spring elements**

#### **6 Gap elements**

#### **7 Slippings elements**

---

### **2D elements**

#### **1 Plate elements / Shell elements**

- 3Node/ 4nodes are required
- It has 6 DOF ( 3 translational and 3 rotational )

#### **2 Membrane elements**

- 3Node/ 4nodes are required
- Only translational degrees of freedom

---

### **3D elements**

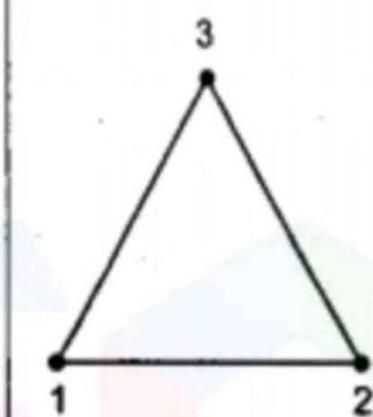
#### **1 Tetrahedral / hexa Element**

- 4 nodes / 5nodes / 6nodes / 7nodes / 8nodes / 15nodes / 20 nodes
- Only translational degree of freedom at each node

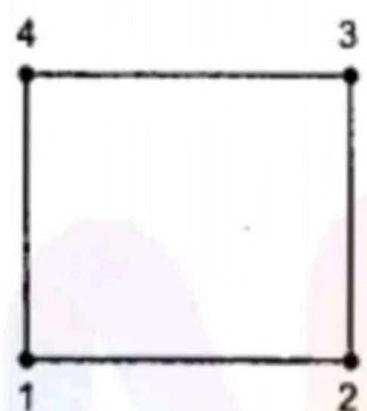
# Q. What are the stages for finite elements analysis ?

Discuss in details .

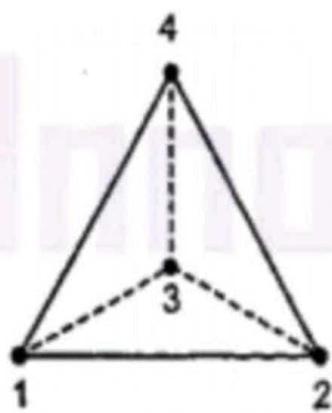
Ans : • The first step in FEA is Discretization or Splitting into multiple smaller elements. This is done depending upon Geometry. This step is also called Meshing. And the elements can be 1D Bar, 2D Triangle and Quad, 3D Tetra and Hex.



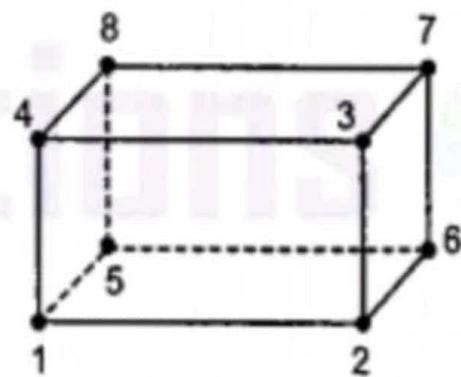
*Triangular element*



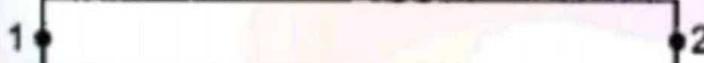
*Rectangular element*



*Tetrahedral element*

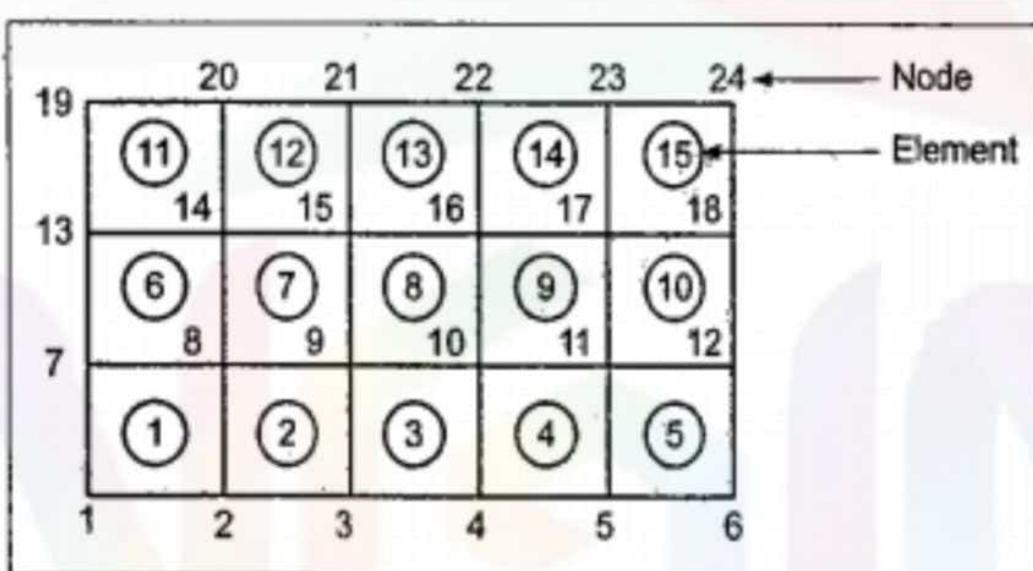


*Hexahedral element*



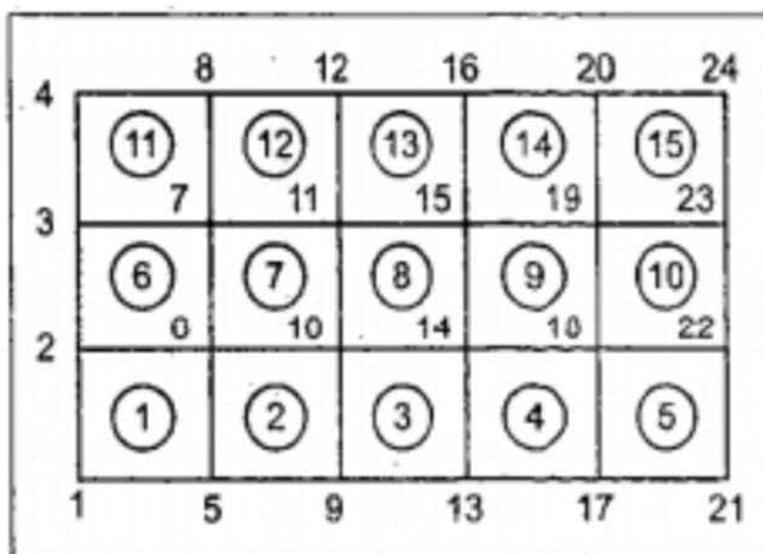
- Second step is Node Numbering. This can be done by Long side numbering process or short side numbering process. FEA will follow the method which obeys  $(\text{Max Node Number} - \text{Min node number}) = \text{Minimum}$  for any particular element to reduce memory requirements.

This is long side numbering.



Here for element 3, Max node number-Min node number =  $10 - 3 = 7$

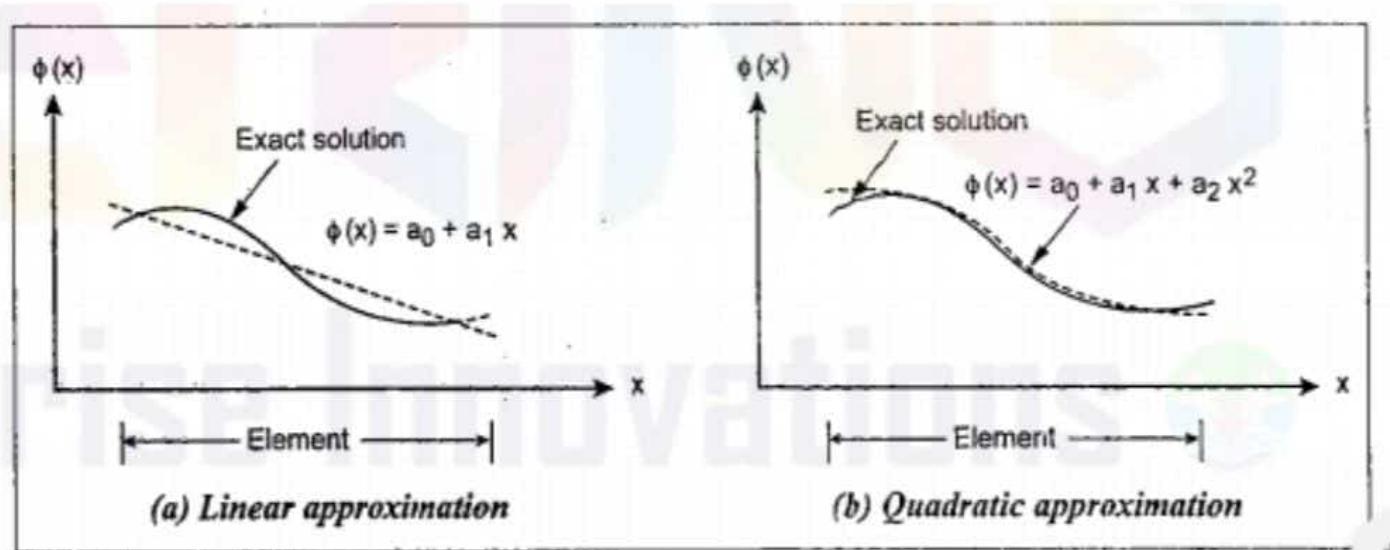
This is short side numbering.



Here for element 3, Max node number-Min node number =  $14 - 9 = 5$ .

Hence, short side numbering is preferred.

- The third step is Selection/Assignment of a Function. Now why do we need to assign a function? A function helps to describe the behavior of the element. It can be a polynomial. Polynomials are used as they can be integrated and differentiated. Also, higher order polynomials are preferred to give good results. And higher order polynomials also resist shear locking.



- Define Material behavior.
- Derivation of Element stiffness matrix for each element.
- Arrive at the Global Stiffness matrix.

- Arrive at the Global Stiffness matrix.

$$\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ \vdots \\ F_n \end{array} \right\} = \left[ \begin{array}{cccc} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \dots & k_{4n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_n \end{array} \right\}$$

- Apply Boundary conditions.
- Solve for unknown displacements. It is to be noted that in FEA, there are two basic methods - the Force and the Displacement method. Normally we prefer the latter and solve for displacements as it is simpler.
- Computation of Strains and Stress.

**Q. Write shortnotes on 'principle of minimum potential energy'.**

### Principle of Minimum Potential Energy:

- Deformation and stress analysis of structural system and elastic analysis of structures can be accomplished by using the principle of minimum potential (MPE).
- It states that for conservative structure system, of all the kinematically admissible deformations, those corresponding to the equilibrium state extremize (i.e., minimize or maximize) the total potential energy. If the extremum is a minimum, the equilibrium state is stable.

### STEPS:

- Differentiate the total potential energy ( $\Pi$ ) of an elastic structure w.r.t. nodal displacements  $q_i$ ,  $i = 1, 2, 3$  ( $\leftarrow$  node points) and then equate to zero.

$$\text{i.e., } \frac{\partial \Pi}{\partial q_i} = 0$$

- Write /arrange above equ<sup>n</sup>. in the matrix form

$$[K][q] = [F]$$

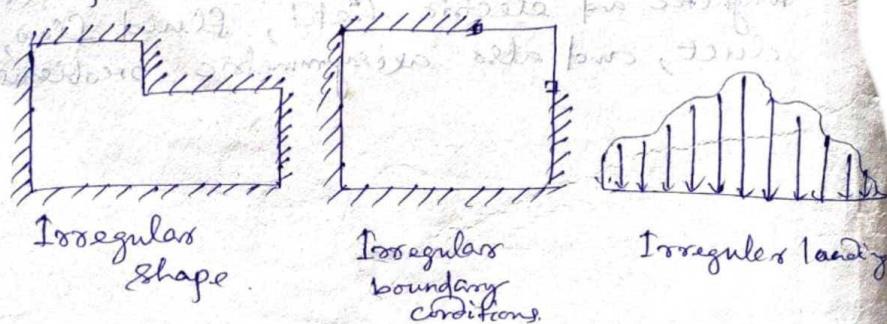
and solve for nodal displacements  $q_1, q_2, q_3 \dots$

## Q. Mention important features of FEM.

1. Solution have been obtained for few standard cases by classical method, whereas solution can be obtained for all problems by finite element analysis.
2. Whenever the following complexities are faced classical method makes the drastic assumption and looks for the solutions:  
① Shape, ② Boundary conditions ③ Loading.  
Figure shows such cases in the analysis of Slabs (plates).

To get the solutions in the above cases, rectangular shapes, same boundary condition along a side and regular equivalent loads are to be assumed.

In FEM, no such assumptions are made. The problem is treated as it is.



3. When material properties is not isotropic, solutions for the problem becomes very difficult in solving in classical method. Only few simple cases have been tried successfully by researchers.

FEM can handles structures with anisotropic properties also without any difficulty.

4. If structures consists of more than one material, it is difficult to use classical method but finite element can be used without any difficult.
5. Problems with material and geometric non-linearity can not be handled by classical methods. But there is no difficulty in FEM.

Hence FEM is superior to the classical methods only for the problems involving a number of complexities which can not be handled by classical methods without making drastic assumptions.

- # For all regular problems, the solutions by classical methods are the best solutions in fact.

## Q. Write about Galerkin's Finite Element Method .

### Ans : Galerkin's Finite Element Method :

In order to obtain a numerical solution to a differential equation using the Galerkin Finite Element Method (GFEM), the domain is subdivided into **finite elements**. The function is approximated by piecewise **trial functions** over each of these elements. This is illustrated below for the one-dimensional case, with *linear* functions used over each element,  $p$  being the dependent variable.

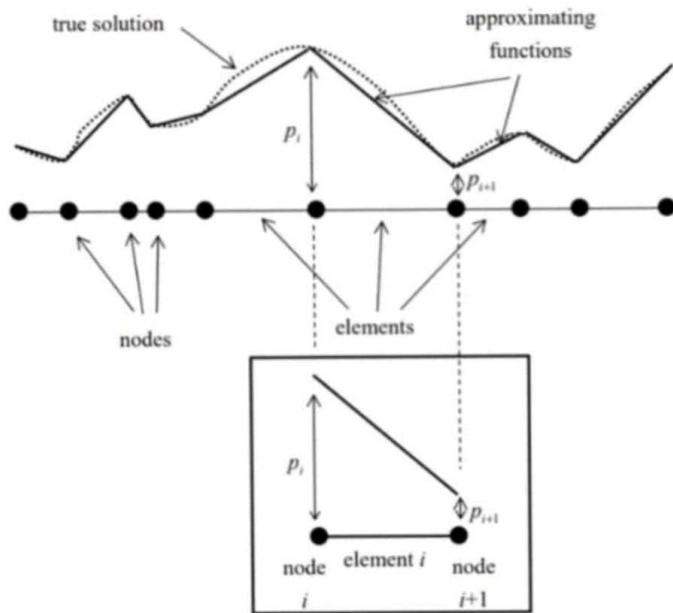


Figure 2.1: A mesh of  $N$  one dimensional Finite Elements

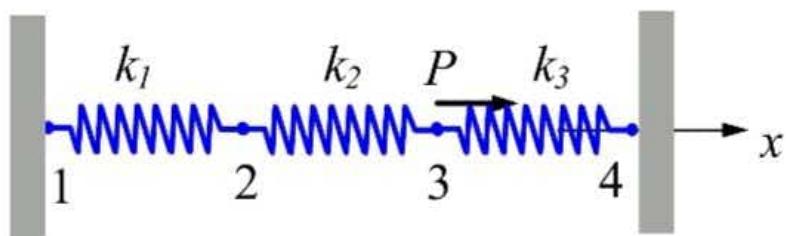
The unknowns of the problem are the **nodal values** of  $p$ ,  $p_i$ ,  $i = 1 \dots N + 1$ , at the **element boundaries** (which in the 1D case are simply points). The (approximate) solution within each element can then be constructed once these nodal values are known.

The Galerkin FEM for the solution of a differential equation consists of the following steps:

- (1) multiply the differential equation by a **weight function**  $\omega(x)$  and form the integral over the whole domain
- (2) if necessary, integrate by parts to reduce the order of the highest order term
- (3) choose the order of interpolation (e.g. linear, quadratic, etc.) and corresponding shape functions  $N_i$ ,  $i = 1 \dots m$ , with trial function  $p = \tilde{p}(x) = \sum_{i=1}^m N_i(x)p_i$
- (4) evaluate all integrals over each element, either exactly or numerically, to set up a system of equations in the unknown  $p_i$ 's
- (5) solve the system of equations for the  $p_i$ 's.

The linear  $C^0$  element will be used in what follows. Quadratic and cubic elements will be considered later.

### Example 1.1



*Given:* For the spring system shown above,

$$k_1 = 100 \text{ N/mm}, \quad k_2 = 200 \text{ N/mm}, \quad k_3 = 100 \text{ N/mm}$$

$$P = 500 \text{ N}, \quad u_1 = u_4 = 0$$

*Find:* (a) the global stiffness matrix

(b) displacements of nodes 2 and 3

(c) the reaction forces at nodes 1 and 4

(d) the force in the spring 2

**Solution:**

(a) The element stiffness matrices are

$$\mathbf{k}_1 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \text{ (N/mm)} \quad (1)$$

$$\mathbf{k}_2 = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \text{ (N/mm)} \quad (2)$$

$$\mathbf{k}_3 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \text{ (N/mm)} \quad (3)$$

Applying the superposition concept, we obtain the global stiffness matrix for the spring system as

$$\mathbf{K} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ \hline 100 & -100 & 0 & 0 \\ -100 & 100+200 & -200 & 0 \\ 0 & -200 & 200+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}$$

or

$$\mathbf{K} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}$$

which is *symmetric* and ***banded***.

Equilibrium (FE) equation for the whole system is

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \\ F_4 \end{Bmatrix} \quad (4)$$

(b) Applying the BC ( $u_1 = u_4 = 0$ ) in Eq(4), or deleting the 1<sup>st</sup> and 4<sup>th</sup> rows and columns, we have

$$\begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \quad (5)$$

Solving Eq.(5), we obtain

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P/250 \\ 3P/500 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \text{ (mm)} \quad (6)$$

(c) From the 1<sup>st</sup> and 4<sup>th</sup> equations in (4), we get the reaction forces

$$F_1 = -100u_2 = -200 \text{ (N)}$$

$$F_4 = -100u_3 = -300 \text{ (N)}$$

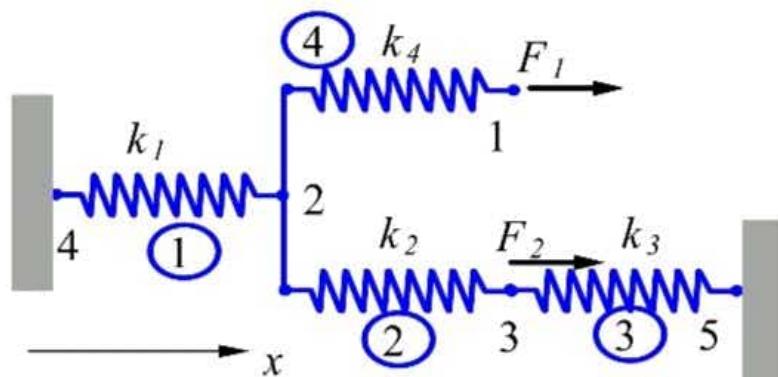
(d) The FE equation for spring (element) 2 is

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Here  $i = 2, j = 3$  for element 2. Thus we can calculate the spring force as

$$\begin{aligned} F = f_j = -f_i &= [-200 \quad 200] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \\ &= [-200 \quad 200] \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \\ &= 200 \text{ (N)} \end{aligned}$$

### Example 1.2



**Problem:** For the spring system with arbitrarily numbered nodes and elements, as shown above, find the global stiffness matrix.

**Solution:**

First we construct the following

**Element Connectivity Table**

<i>Element</i>	<i>Node i (1)</i>	<i>Node j (2)</i>
1	4	2
2	2	3
3	3	5
4	2	1

which specifies the *global* node numbers corresponding to the *local* node numbers for each element.

Then we can write the element stiffness matrices as follows

$$\mathbf{k}_1 = \begin{bmatrix} u_4 & u_2 \\ k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \quad \mathbf{k}_2 = \begin{bmatrix} u_2 & u_3 \\ k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

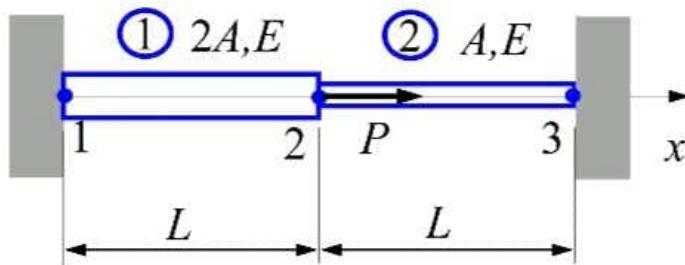
$$\mathbf{k}_3 = \begin{bmatrix} u_3 & u_5 \\ k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \quad \mathbf{k}_4 = \begin{bmatrix} u_2 & u_1 \\ k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

Finally, applying the superposition method, we obtain the global stiffness matrix as follows

$$\mathbf{K} = \left[ \begin{array}{c|cc|cc|c} u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline k_4 & -k_4 & 0 & 0 & 0 \\ -k_4 & k_1 + k_2 + k_4 & -k_2 & -k_1 & 0 \\ \hline 0 & -k_2 & k_2 + k_3 & 0 & -k_3 \\ 0 & -k_1 & 0 & k_1 & 0 \\ \hline 0 & 0 & -k_3 & 0 & k_3 \end{array} \right]$$

The matrix is *symmetric, banded, but singular.*

### Example 2.1



*Problem:* Find the stresses in the two bar assembly which is loaded with force  $P$ , and constrained at the two ends, as shown in the figure.

*Solution:* Use two 1-D bar elements.

Element 1,

$$\mathbf{k}_1 = \frac{2EA}{L} \begin{bmatrix} u_1 & u_2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 2,

$$\mathbf{k}_2 = \frac{EA}{L} \begin{bmatrix} u_2 & u_3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Imagine a frictionless pin at node 2, which connects the two elements. We can assemble the global FE equation as follows,

$$\frac{EA}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \quad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1<sup>st</sup> row and column, and the 3<sup>rd</sup> row and column, we obtain,

$$\frac{EA}{L} [3] \{u_2\} = \{P\}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Stress in element 1 is

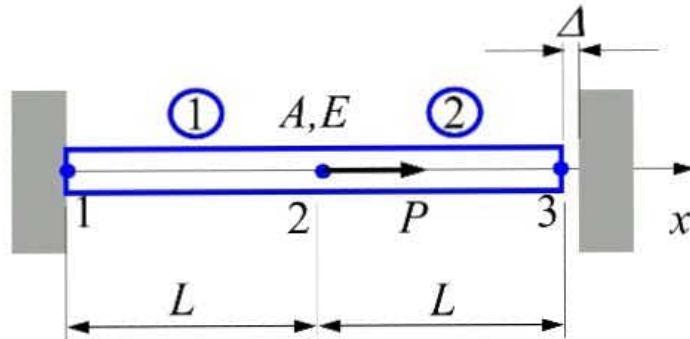
$$\begin{aligned}\sigma_1 &= E\epsilon_1 = E\mathbf{B}_1 \mathbf{u}_1 = E[-1/L \quad 1/L] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= E \frac{u_2 - u_1}{L} = \frac{E}{L} \left( \frac{PL}{3EA} - 0 \right) = \frac{P}{3A}\end{aligned}$$

Similarly, stress in element 2 is

$$\begin{aligned}\sigma_2 &= E\epsilon_2 = E\mathbf{B}_2 \mathbf{u}_2 = E[-1/L \quad 1/L] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \\ &= E \frac{u_3 - u_2}{L} = \frac{E}{L} \left( 0 - \frac{PL}{3EA} \right) = -\frac{P}{3A}\end{aligned}$$

which indicates that bar 2 is in compression.

### Example 2.2



*Problem:* Determine the support reaction forces at the two ends of the bar shown above, given the following,

$$P = 6.0 \times 10^4 \text{ N}, \quad E = 2.0 \times 10^4 \text{ N/mm}^2,$$

$$A = 250 \text{ mm}^2, \quad L = 150 \text{ mm}, \quad \Delta = 1.2 \text{ mm}$$

*Solution:*

We first check to see if or not the contact of the bar with the wall on the right will occur. To do this, we imagine the wall on the right is removed and calculate the displacement at the right end,

$$\Delta_0 = \frac{PL}{EA} = \frac{(6.0 \times 10^4)(150)}{(2.0 \times 10^4)(250)} = 1.8 \text{ mm} > \Delta = 1.2 \text{ mm}$$

Thus, contact occurs.

The global FE equation is found to be,

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

The load and boundary conditions are,

$$F_2 = P = 6.0 \times 10^4 \text{ N}$$

$$u_1 = 0, \quad u_3 = \Delta = 1.2 \text{ mm}$$

FE equation becomes,

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ \Delta \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

The 2<sup>nd</sup> equation gives,

$$\frac{EA}{L} [2 \quad -1] \begin{Bmatrix} u_2 \\ \Delta \end{Bmatrix} = \{P\}$$

that is,

$$\frac{EA}{L} [2] \{u_2\} = \left\{ P + \frac{EA}{L} \Delta \right\}$$

Solving this, we obtain

$$u_2 = \frac{1}{2} \left( \frac{PL}{EA} + \Delta \right) = 1.5 \text{ mm}$$

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.5 \\ 1.2 \end{Bmatrix} \text{ (mm)}$$

To calculate the support reaction forces, we apply the 1<sup>st</sup> and 3<sup>rd</sup> equations in the global FE equation.

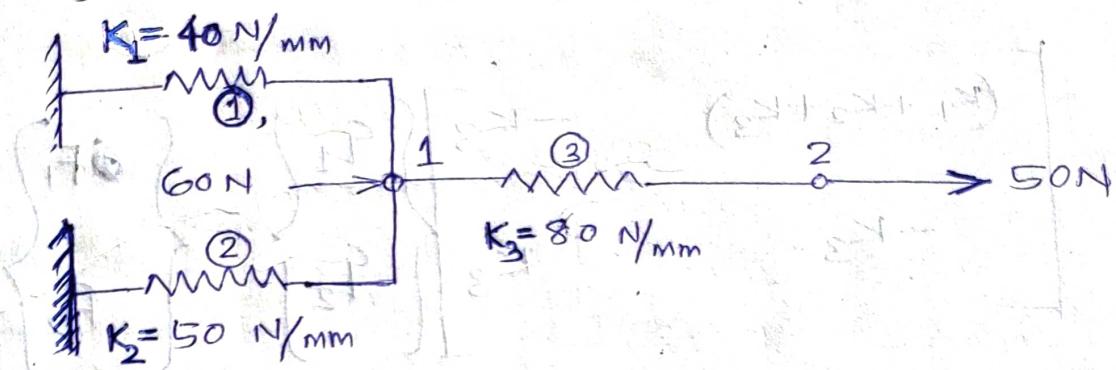
The 1<sup>st</sup> equation gives,

$$F_1 = \frac{EA}{L} [1 \quad -1 \quad 0] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L} (-u_2) = -5.0 \times 10^4 \text{ N}$$

and the 3<sup>rd</sup> equation gives,

$$\begin{aligned} F_3 &= \frac{EA}{L} [0 \quad -1 \quad 1] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L} (-u_2 + u_3) \\ &= -1.0 \times 10^4 \text{ N} \end{aligned}$$

Problem-1: Determine the displacements of nodes of the spring system shown in following fig.



Solution: Let  $\delta_1, \delta_2$  and  $\delta_3$  are the extensions of springs.  $q_1$  and  $q_2$  are the displacements of the nodes 1 and 2.

Then

$$\delta_1 = q_1, \quad \delta_2 = q_1, \quad \delta_3 = q_2 - q_1 \quad F_1 = 60 \text{ N}$$

$$K_1 = 40 \text{ N/mm}, \quad K_2 = 50 \text{ N/mm}, \quad K_3 = 80 \text{ N/mm}, \quad F_2 = 50 \text{ N}$$

For the given spring system, the total potential energy will be

$$\Pi = \frac{1}{2} K_1 \delta_1^2 + \frac{1}{2} K_2 \delta_2^2 + \frac{1}{2} K_3 \delta_3^2 - F_1 q_1 - F_2 q_2$$

$$= \frac{1}{2} K_1 q_1^2 + \frac{1}{2} K_2 q_1^2 + \frac{1}{2} K_3 (q_2 - q_1)^2 - F_1 q_1 - F_2 q_2$$

By applying minimum potential principle,

$$\frac{\partial \Pi}{\partial q_i} = 0, \quad i = 1, 2$$

$$\frac{\partial \Pi}{\partial q_1} = k_1 q_1 + k_2 q_1 - k_3(q_2 - q_1) - F_1 = 0$$

$$\Rightarrow k_1 q_1 + k_2 q_1 - k_3 q_2 + k_3 q_1 = F_1$$

$$\Rightarrow (k_1 + k_2 + k_3) q_1 - k_3 q_2 = F_1 \quad \text{--- (1)}$$

$$\frac{\partial \Pi}{\partial q_2} = k_3(q_2 - q_1) - F_2 = 0$$

$$\Rightarrow -k_3 q_1 + k_3 q_2 = F_2 \quad \text{--- (2)}$$

In matrix form, equ's (1) & (2) can be written as

$$\begin{bmatrix} (k_1 + k_2 + k_3) & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} (40+50+80) & -80 \\ -80 & 80 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 50 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 170 & -80 \\ -80 & 80 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 50 \end{Bmatrix}$$

$$170 q_1 - 80 q_2 = 60$$

$$-80 q_1 + 80 q_2 = 50$$

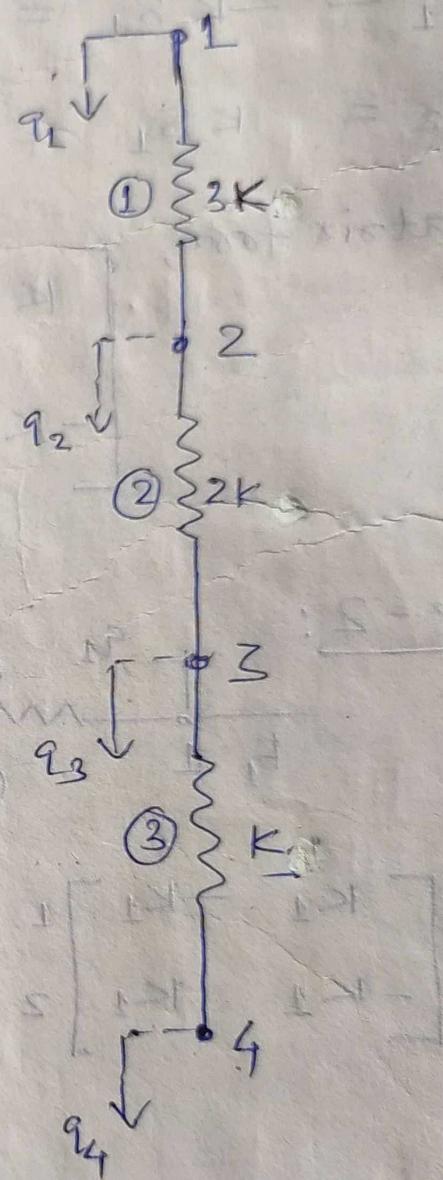
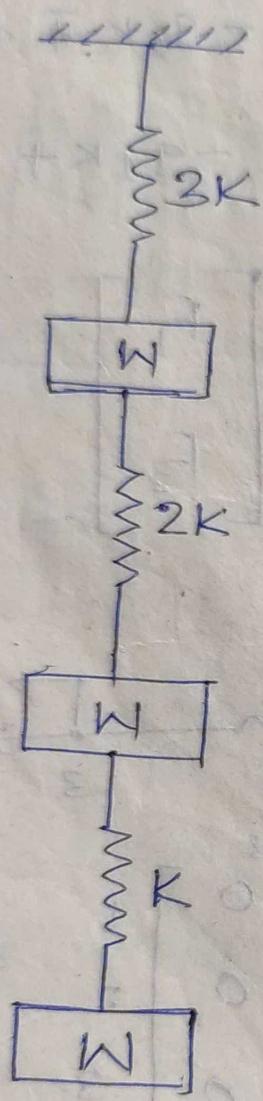
By solving above two linear equations, we get

$$q_1 = 1.221 \text{ mm}$$

$$q_2 = 1.846 \text{ mm}$$

Ans

Example - 3 Determine the support reaction 'R' at the support.



The element stiffness matrices will be,

$$K_1 = \begin{bmatrix} 1 & 2 \\ 3K & -3K \\ -3K & 3K \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2 & 3 \\ 2K & -2K \\ -2K & 2K \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 3 & 4 \\ K & -K \\ -K & K \end{bmatrix}$$

The global stiffness matrix will be,

$$K = K_1 + K_2 + K_3$$

$$\Rightarrow K = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3K & -3K & 0 & 0 \\ -3K & (3K+2K) & -2K & 0 \\ 0 & -2K & (2K+K) & -K \\ 0 & 0 & -K & K \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3K & -3K & 0 & 0 \\ -3K & 5K & -2K & 0 \\ 0 & -2K & 3K & -K \\ 0 & 0 & -K & K \end{bmatrix}$$

The solve  $K q = F$

$$\Rightarrow \begin{bmatrix} 3K & -3K & 0 & 0 \\ -3K & 5K & -2K & 0 \\ 0 & -2K & 3K & -K \\ 0 & 0 & -K & K \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix}$$

\* Always take "zero displacement" for fixed node

$$\begin{bmatrix} 5K & -2K & 0 \\ -2K & 3K & -K \\ 0 & -K & K \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

$$\Rightarrow K \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{1}{K} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} W \\ W \\ W \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{1}{K} \begin{bmatrix} 0.333 & 0.333 & 0.333 \\ 0.333 & 0.833 & 0.833 \\ 0.333 & 0.833 & 1.833 \end{bmatrix} \begin{Bmatrix} W \\ W \\ W \end{Bmatrix}$$

$$\Rightarrow q_2 = \frac{1}{K} \left[ \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right] W = \frac{W}{K}$$

$$q_2 = \frac{1}{K} \left[ \frac{1}{3} + 0.833 + 0.833 \right] W = \frac{2W}{K}$$

$$q_4 = \frac{1}{K} \left[ \frac{1}{3} + 0.833 + 1.833 \right] W = \frac{3W}{K}$$

Unknown reaction at the fixed node -1, will be

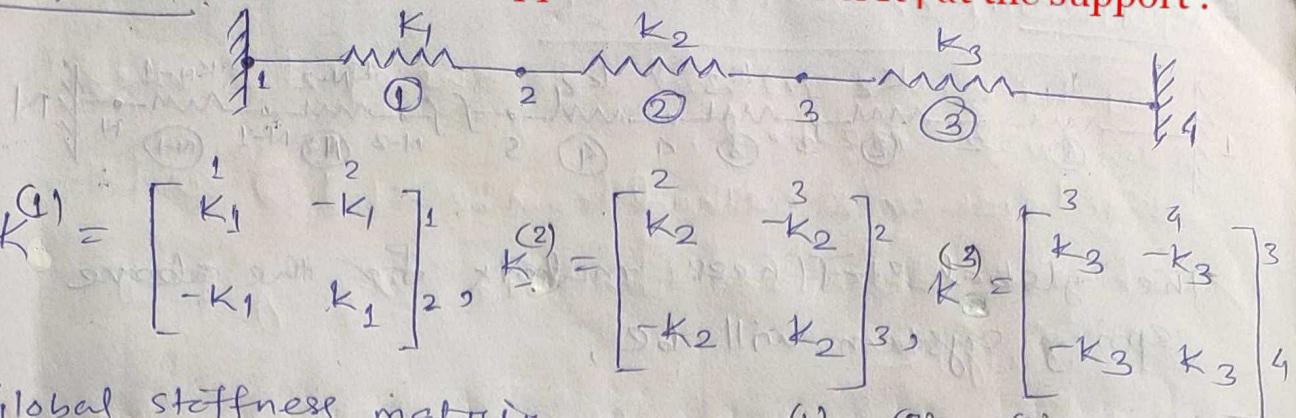
$$R = \begin{bmatrix} 3K & -3K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\Rightarrow R = 3Kq_1 - 3Kq_2$$

$$\text{As } q_1 = 0, \quad R = -3Kq_2 = -3K \frac{W}{K}$$

$$\Rightarrow W \boxed{R = -3W}$$

Example -4 Determine the support reaction R<sub>1</sub> & R<sub>4</sub> at the support.



$$K^{(1)} = \begin{bmatrix} 1 & 2 \\ K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix}_1, \quad K^{(2)} = \begin{bmatrix} 2 & 3 \\ K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}_2, \quad K^{(3)} = \begin{bmatrix} 3 & 4 \\ K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix}_3$$

Global stiffness matrix,  $K = K^{(1)} + K^{(2)} + K^{(3)}$

$$K = \begin{bmatrix} 1 & 2 & 3 & 4 \\ K_1 & -K_1 & 0 & 0 \\ -K_1 & (K_1+K_2) & -K_2 & 0 \\ 0 & -K_2 & (K_2+K_3) & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix}$$

Now solve  $-Kq = F$

$$\text{or, } \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & (K_1+K_2) & -K_2 & 0 \\ 0 & -K_2 & (K_2+K_3) & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Here  $q_1$  and  $q_4$  are zero, then the modified system of matrix will be,

$$\begin{bmatrix} (K_1+K_2) & -K_2 \\ -K_2 & (K_2+K_3) \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

Unknown Reactions, R<sub>1</sub> & R<sub>4</sub>

$$R_1 = K_1 q_1 - K_1 q_2 = -K_1 q_2$$

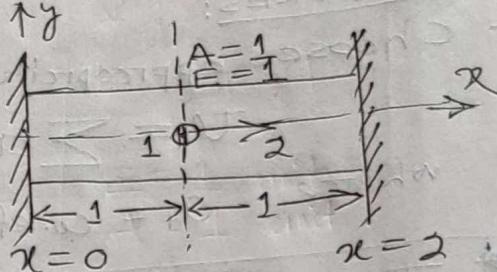
~~$$R_2 = -K_2 q_1 + (K_1+K_2) q_2 - K_2 q_3$$~~

$$R_4 = -K_3 q_3 + K_3 q_4 = -K_3 q_3$$

Problem-1: The potential energy for the elastic 1-D bar, shown in fig, is given by

$$\Pi = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 dx - 2\ell u_1, \text{ where } u_1 = u \text{ at } x=1$$

Find out the displacement field  $u(x)$  and Stress  $\sigma(x)$ .



Solution: Given

Potential energy,  $\Pi = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 dx - 2\ell u_1$   
where  $u_1 = u \text{ at } x=1$ .

Here  $m = 2$

$$n = m+1 = 2+1 = 3$$

Let us consider a polynomial function for displacement,  $u = a_1 + a_2 x + a_3 x^2$

For boundary cond<sup>n</sup>,  $x=0, u=0$

$$\boxed{a_1 = 0}$$

Again for  $x=2, u=0$

$$2a_2 + 4a_3 = 0$$

$$\Rightarrow \boxed{a_2 = -2a_3}$$

Now

$$\frac{du}{dx} = a_2 + 2a_3 x$$

Now  $\Pi = \frac{1}{2} \int_0^2 EA \left( a_2 + 2a_3 x \right)^2 dx -$

For  $u_1 = u \text{ at } x=1$

Now,  $u = -2a_3 x + a_3 x^2$

$$\frac{du}{dx} = -2a_3 + 2a_3 x \quad \text{--- (1)}$$

$$\text{For } u_1 = \ell (x=1)$$

$$\text{i.e. } u_1 = -2\alpha_3 + \alpha_3$$

$$\Rightarrow \boxed{u_1 = -\alpha_3}$$

$$\therefore \Pi = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 dx - 2u_1$$

$$\Rightarrow \Pi = \frac{1}{2} \int_0^L 1 \times 1 (-2\alpha_3 + 2\alpha_3 x)^2 dx + 2\alpha_3$$

$$= \frac{1}{2} \int_0^L (4\alpha_3^2 - 8\alpha_3^2 x + 4\alpha_3^2 x^2) dx + 2\alpha_3$$

$$= \frac{1}{2} \left[ 4\alpha_3^2 x - 4\alpha_3^2 x^2 + \frac{4}{3}\alpha_3^2 x^3 \right]_0^L + 2\alpha_3$$

$$= \frac{1}{2} \left( 8\alpha_3^2 - 16\alpha_3^2 + \frac{32}{3}\alpha_3^2 \right) + 2\alpha_3$$

$$= \frac{4}{3}\alpha_3^2 + 2\alpha_3$$

$$\Rightarrow \Pi = \frac{4}{3}\alpha_3^2 + 2\alpha_3 \quad \underline{\text{(ii)}}$$

$$\text{Now } \frac{d\Pi}{d\alpha_3} = \frac{8}{3}\alpha_3 + 2 = 0$$

$$\Rightarrow \boxed{\alpha_3 = -\frac{3}{4} = -0.75 \text{ m}}$$

Hence the displacement,  $u = -\frac{3}{4}(-2x+x^2)$

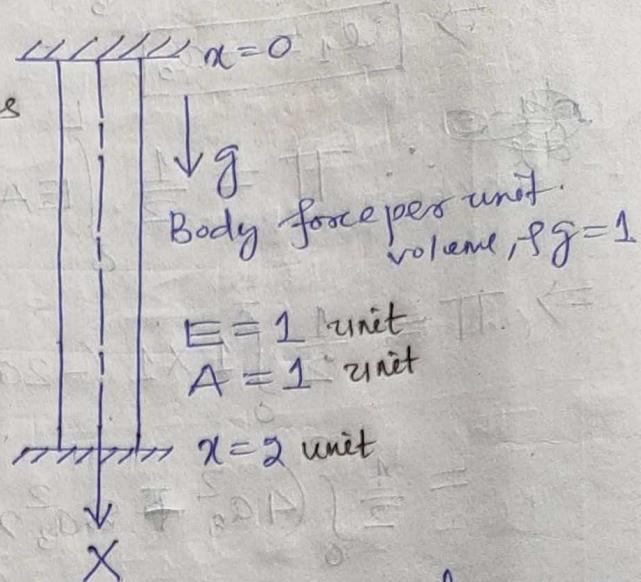
$$\Rightarrow u = 1.5x - 0.75x^2$$

$$\sigma = E\epsilon = E \frac{du}{dx} = 1 \times \left[ -2x(-\frac{3}{4}) + 2(-\frac{3}{4})x \right]$$

$$\Rightarrow \sigma = 1.5 - 1.5x \text{ my}$$

Problem-2: Use the Rayleigh-Ritz method to find the displacement of the mid-point of the rod shown in following figure.

Also, determine the stress at the mid-point of the rod.



Solution: The expression for the total potential energy,

$$\Pi = \text{Strain energy} - \text{Work potential}$$

$$\text{Strain energy, } = \frac{1}{2} \int \sigma^T \epsilon dV$$

$$\epsilon = \frac{du}{dx}, \sigma = E\epsilon = E \frac{du}{dx}, V = Adx$$

$$\therefore S.E. = \frac{1}{2} \int_V EA \left( \frac{du}{dx} \right)^2 dx$$

$$\text{Body force, } f = \rho g = 1$$

$$\text{Work potential} = - \int_V u e(f dV) = \int_V u f A dx$$

$$\text{Hence } \Pi = \frac{1}{2} \int_V EA \left( \frac{du}{dx} \right)^2 dx - \int_V u f A dx$$

$$\text{Let us take } m=2, m=m+1=2+1=3$$

Let us consider the polynoimial for

$$u = a_1 + a_2 x + a_3 x^2$$

$$\text{For boundary condition, } x=0, u=0$$

$$\therefore [a_1 = 0]$$

$$\text{For B.C. } x=2, u=0$$

$$2a_2 + 4a_3 = 0 \Rightarrow [a_2 = -2a_3]$$

By substituting,

$$U = -2a_3x + a_3x^2 \quad \text{--- (1)}$$

$$\frac{dU}{dx} = -2a_3 + 2a_3x$$

$$\therefore \Pi = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 dx - \int_0^L U f A dx$$

$$= \frac{1}{2} \int_0^L EA (-2a_3 + 2a_3x)^2 dx - \int_0^L (-2a_3x + a_3x^2) f A dx$$

$$= \frac{1}{2} \int_0^L (4a_3^2 - 8a_3^2 x + 4a_3^2 x^2) dx - \int_0^L (-2a_3x + a_3x^2) dx$$

$$= \frac{1}{2} \left[ 4a_3^2 x - 4a_3^2 x^2 + \frac{4}{3} a_3^2 x^3 \right]_0^2 - \left[ -2a_3x^2 + \frac{a_3x^3}{3} \right]_0^2$$

$$= \frac{1}{3} \left[ 8a_3^2 - 16a_3^2 + \frac{32}{3} a_3^2 \right] - \left[ -4a_3 - \frac{8a_3}{3} \right]$$

$$\boxed{\Pi = \frac{4}{3} a_3^2 + \frac{4}{3} a_3}$$

$$\text{Now } \frac{d\Pi}{da_3} = \frac{4}{3} \times 2a_3 + \frac{4}{3} = 0 \quad \text{--- (2)}$$

$$\Rightarrow \boxed{a_3 = -\frac{1}{2}}$$

$$\text{Hence, } U = x - \frac{x^2}{2} \quad \text{--- (1)}$$

displacement of mid point can be obtained by substituting,  $x = 1$  in eqn (1),

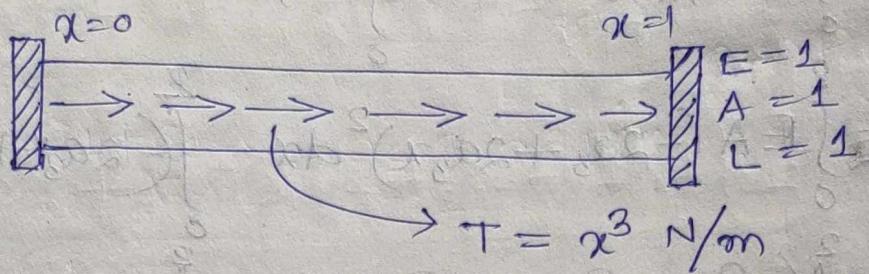
$$\boxed{U = 0.5}$$

Stress Field,  $\sigma(x)$  at the mid point of rod :

$$\sigma(x) = E \cdot \epsilon = E \cdot \frac{du}{dx} = 1 \times (-2a_3 + 2a_3x) \Big|_{a_3 = -\frac{1}{2}}$$

$$\Rightarrow \boxed{\sigma(x) = 1 - x} \quad \therefore \sigma(x) \Big|_{x=1} = 1 - x \Big|_{(x=1)} = 1 - 1 = 0$$

Problem :- 3: A rod fixed at its ends is subjected to a traction force as shown. Use the Rayleigh-Ritz method with an assumed displacement field  $u = a_0 + a_1x + a_2x^2$  to determine displacement  $u(x)$  and stress  $\sigma(x)$ .



Solution: The expression for total potential energy is,

$$T = \text{Strain energy} + \text{Work potential}$$

$$\text{Strain energy} = \frac{1}{2} \int_0^L \sigma \epsilon \, dx = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 \, dx$$

$$\text{Work potential} = - \int_0^L T \, dx - \int_0^L u T \, dx$$

$$\therefore T = \frac{1}{2} \int_0^L EA \left( \frac{du}{dx} \right)^2 \, dx - \int_0^L T \, dx$$

Given  $u = a_0 + a_1x + a_2x^2$

For boundary condition,  $x=0, u=0$

$$\boxed{a_0 = 0} \quad x = 0 \quad \text{result}$$

For boundary condition,  $x=L, u=0$

$$\textcircled{1} \quad a_1 + a_2 = 0 \quad \text{result}$$

$$\textcircled{2} \quad \boxed{a_1 = -a_2}$$

Hence  $u = -a_2x + a_2x^2$

$$\frac{du}{dx} = -a_2 + 2a_2x$$

$$\begin{aligned}
 \text{Now } \Pi &= \frac{1}{2} \int_0^1 EA(-\alpha_2 + 2\alpha_2 x)^2 dx - \int_0^1 (-\alpha_2 x + \alpha_2 x^2) x^3 dx \\
 &= \frac{1}{2} \int_0^1 (\alpha_2^2 - 4\alpha_2^2 x + 4\alpha_2^2 x^2) dx - \int_0^1 (-\alpha_2 x^4 + \alpha_2 x^5) dx \\
 &= \frac{1}{2} \left[ \alpha_2^2 x - 2\alpha_2^2 x^2 + \frac{4}{3} \alpha_2^2 x^3 \right]_0^1 - \left[ -\frac{\alpha_2 x^5}{5} + \frac{\alpha_2 x^6}{6} \right]_0^1 \\
 &= \frac{1}{2} \left[ \alpha_2^2 - 2\alpha_2^2 + \frac{4}{3} \alpha_2^2 \right] - \left[ \frac{\alpha_2}{5} - \frac{\alpha_2}{6} \right]
 \end{aligned}$$

$$\Rightarrow \boxed{\Pi = \frac{1}{6} \alpha_2^2 + \frac{1}{30} \alpha_2}$$

$$\therefore \frac{d\Pi}{d\alpha_2} = \frac{1}{3} \alpha_2 + \frac{1}{30} = 0$$

$$\Rightarrow \boxed{\alpha_2 = -\frac{1}{10} = -0.1}$$

displacement field,  $u_e = -0.1x + 0.1x^2$

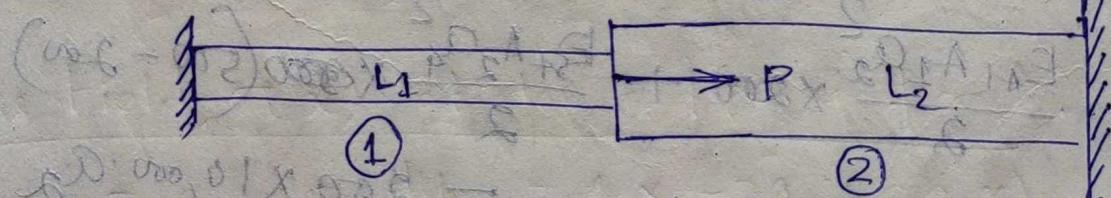
Stress,  $\sigma = E\varepsilon = E \frac{du}{dx} = 0.1 - 0.2x$

Problem 4: Use the Rayleigh-Ritz method to find the displacement field  $u(w)$  of the rod shown in figure. Element 1 is made of aluminum, and element 2 is made of steel. The properties are

$E_{Al} = 70 \text{ GPa}$ ,  $A_1 = 900 \text{ mm}^2$ ,  $L_1 = 200 \text{ mm}$ .

$E_{St} = 200 \text{ GPa}$ ,  $A_2 = 1200 \text{ mm}^2$ ,  $L_2 = 300 \text{ mm}$ .

Load  $P = 10,000 \text{ N}$ . Assume a piecewise linear displacement field  $u_e = \alpha_1 + \alpha_2 x$  for  $0 \leq x \leq 200 \text{ mm}$



and  $u = \alpha_3 + \alpha_4 x$  for  $200 \leq x \leq 500 \text{ mm}$ .

Solution: The ~~solutions~~ expression for potential energy is given by,

$\Pi = \text{Strain energy} + \text{work potential}$

$$= \frac{1}{2} \int_0^{200} E_A \left( \frac{du}{dx} \right)^2 dx + \frac{1}{2} \int_{200}^{500} E_{st} A_2 \left( \frac{du}{dx} \right)^2 dx - P \cdot u_{200}$$

$$u = a_1 + a_2 x, \quad 0 \leq x \leq 200 \text{ mm}$$

$$\text{At } x=20, u=0 \quad \Rightarrow \quad [a_1 + 20a_2 = 0] \quad \boxed{a_1 = 0}$$

$$u = a_2 x \quad \text{--- (1)} \quad , \quad \frac{du}{dx} = a_2 \quad \boxed{\Pi}$$

$$\text{Again, } u = a_3 + a_4 x, \quad 200 \leq x \leq 500 \text{ mm}$$

$$\text{At } x=500, u=0, \quad 0 = a_3 + 500a_4$$

$$\Rightarrow \boxed{a_3 = -500a_4},$$

$$\therefore u = -500a_4 + a_2 x \quad \text{--- (ii)}, \quad \frac{du}{dx} = a_2$$

$$(u_{200})_{st} = (u_{200})_{st} \text{ at } x = 200 \text{ mm}$$

$$\Rightarrow 200a_2 = -500a_4 + 200a_4$$

$$\Rightarrow \boxed{a_2 = -\frac{3}{2}a_4} \quad \text{--- (iii)}$$

$$\therefore \Pi = \frac{1}{2} \int_0^{200} E_A A_1 a_2^2 dx + \frac{1}{2} \int_{200}^{500} E_{st} A_2 a_4^2 dx$$

$$= \frac{E_A A_1 a_2^2}{2} \times 200 + \frac{E_{st} A_2 a_4^2}{2} \times (500 - 200)$$

$$- 200 \times 10,000 \cdot a_2$$

$$= \frac{70 \times 10^9 \times 900 \times 10^{-6} \times 200}{2} a_2^2 + \frac{200 \times 10^9 \times 1200 \times 10^{-6} \times 300}{2} \times \left(-\frac{2}{3}\right) a_2^2$$

$$- 200 \times 10,000 a_2$$

$$= 63 \times 10^8 a_2^2 - 2.4 \times 10^{10} a_2^2 - 2 \times 10^6 a_2$$

$$\Rightarrow \Pi = -1.7702 \times 10^{10} a_2^2 - 2 \times 10^6 a_2$$

$$\text{Now } \frac{\partial \Pi}{\partial a_2} = -1.7702 \times 10^{10} - 3.54 \times 10^{10} a_2 - 2 \times 10^6$$

$$\Rightarrow a_2 = -\frac{2 \times 10^6}{3.54 \times 10^{10}} = -0.00056$$

$$\therefore u = -0.00056 x, 0 \leq x \leq 200 \text{ mm}$$

$$\text{Again } (u_{200})_{A1} = (u_{200})_{St} \text{ at } x = 200 \text{ mm}$$

$$\Rightarrow 500 a_4 + 200 a_2 = u_{200}$$

$$\Rightarrow \text{Again } \Pi = \frac{1}{2} \int E_A A_1 a_2^2 dx + \frac{1}{2} \int E_{St} A_2 a_4^2 dx - (-300 a_2) P$$

$$= \frac{70 \times 10^9 \times 900 \times 10^{-6} \times 200}{2} a_2^2$$

$$+ \frac{200 \times 10^9 \times 1200 \times 10^{-6} \times 300}{2} a_4^2 + (300 \times 1000) a_2$$

$$\frac{70 \times 10^9 \times 900 \times 10^{-6} \times 200}{2} \times \left(\frac{3}{2} a_4\right)^2$$

$$\left. \frac{200 \times 10^9 \times 1200 \times 10^{-6} \times 300}{2} a_4^2 + 3 \times 10^6 \times \left(\frac{3}{2} a_4\right)^2 \right)$$

$$\therefore \Pi = 1.41 \times 10^{10} a_4^2 + 3.6 \times 10^{10} a_4^2 - 4.5 \times 10^6 a_4$$

$$\therefore \Pi = 5 \times 10^{10} a_4^2 - 4.5 \times 10^6 a_4$$

$$\frac{\partial \Pi}{\partial a_4} = 10^{11} a_4 - 4.5 \times 10^6 \approx 0$$

$$\Rightarrow \alpha_4 = \frac{4.5 \times 10^6}{10^{11}} \quad \checkmark$$

$$\therefore \alpha_4 = 4.5 \times 10^{-5}$$

$$\therefore u = -\frac{500 \times 4.5 \times 10^{-5}}{22000} + 200(4.5 \times 10^{-5})$$

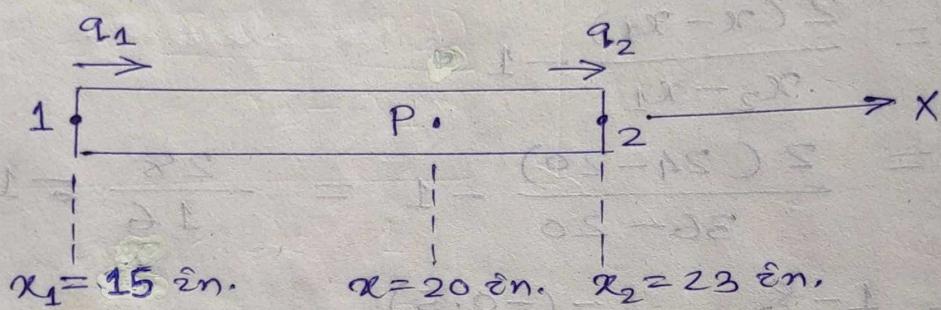
$$u = -500 \times 4.5 \times 10^{-5} + 4.5 \times 10^{-5} x$$

$$\Rightarrow u = 4.5 \times 10^{-5} x - 2250 \times 10^{-5}$$

$$\text{and } x \geq 0, \quad 200 \leq u \leq 500$$

Problem-5: Consider the bar as shown in the following figure. Cross-sectional-area  $A_e = 1.2 \text{ in}^2$ , and Young's modulus  $E = 30 \times 10^6 \text{ psi}$ . If  $q_1 = 0.02 \text{ inch}$  and  $q_2 = 0.025 \text{ inch}$ , determine the following :

- the displacement at point P.
- the strain  $\epsilon$  and stress  $\sigma$
- the element stiffness matrix, and
- the strain energy in the element.



Solution: (a) The displacement at Point 'P'

$$d_p = \frac{2(x - x_1)}{x_2 - x_1} - 1 = \frac{2(20 - 15)}{23 - 15} - 1 = 0.25$$

$$N_1 = \frac{1 - d}{2} = \frac{1 - 0.25}{2} = 0.375$$

$$N_2 = \frac{1 + d}{2} = \frac{1 + 0.25}{2} = 0.625$$

The displacement at point P,  $d_p = N_1 q_1 + N_2 q_2$

$$\Rightarrow d_p = (0.375 \times 0.02) + (0.625 \times 0.025)$$

$$\Rightarrow d_p = 0.0231 \text{ inch.} \quad \boxed{\text{Ans}}$$

(b) Strain,  $\epsilon = B q$

where strain-displacement matrix,  $B = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix}$

$$\Rightarrow B = \frac{1}{23 - 15} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} -0.125 & 0.125 \end{bmatrix}$$

Now  $\epsilon = [-0.125 \quad 0.125] \begin{Bmatrix} 0.02 \\ 0.025 \end{Bmatrix}$

$$= 0.000625$$

Stress,  $\sigma = E\epsilon = 30 \times 10^6 \times 0.000625$

$$= 18750 \text{ psi} \quad \underline{\text{Ans}}$$

(c) The element stiffness matrix,  $K$

$$K = \frac{EA_0}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{30 \times 10^6 \times 1.2}{(23-15)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 4.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \underline{\text{Ans}}$$

(d) Strain energy in the element,  $U$

$$U = \frac{1}{2} q^T K q$$

$$= \frac{1}{2} [0.02 \quad 0.025]^T \times 4.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.02 \\ 0.025 \end{Bmatrix}$$

$$= \frac{4.5 \times 10^6}{2} [0.02 \quad 0.025]^T \begin{Bmatrix} -0.005 \\ 0.005 \end{Bmatrix}$$

$$= \frac{4.5 \times 10^6}{2} \times 0.000025$$

$$\Rightarrow U = 56.25 \text{ lb-in.} \quad \underline{\text{Ans}}$$

**Q. Write down steps of Rayleigh-Ritz method .**

### RayLeigh-Ritz Method

The Rayleigh-Ritz method is used to find out the displacement field at any point of the specified member.

#### PROCEDURES:

- 1- Choose appropriate displacement field, say,

$$u = \sum a_i \phi_i(x, y, z) \leftarrow \text{Variational form}$$

where  $a_i$  = co-efficients,  $\phi_i$  = basis, usually taken as polynomials.

The function for 'u' is usually taken as

$$u = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^n$$

where  $m = \text{degree of the polynomial}$ .

$$n = m+1$$

- 2- Apply boundary conditions in the expression of 'u' to get the value of co-efficients  $a_1, a_2, a_3, \dots$

- 3- Extremize the variational form.

(For this purpose we take the partial derivative with respect to co-efficient only present finally in the expression of potential energy 'Pi')

- 4- If it is asked to calculate the stress field then,

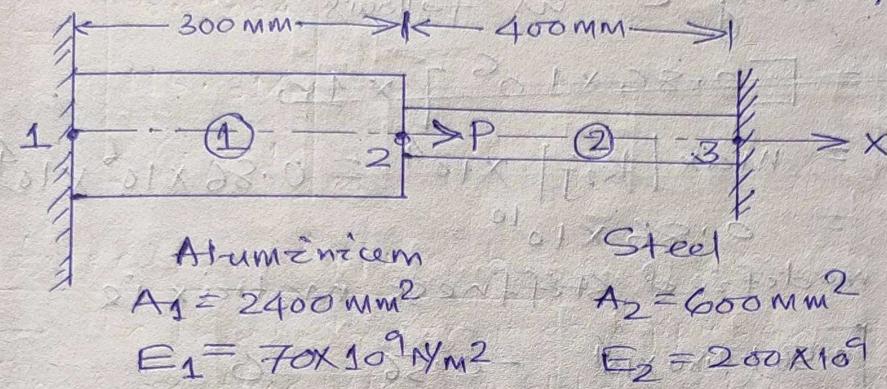
$$\sigma = E \epsilon$$

$$\Rightarrow \sigma = E \frac{du}{dx}$$

where  $u(x)$  is the displacement field

Problem-1: Consider the bar shown in fig. An axial load  $P = 200 \times 10^3 \text{ N}$  is applied as shown. Using the Penalty approach for handling boundary conditions, do the following:

- Determine the nodal displacements.
- Determine the stress in each material.
- Determine the reaction forces.



Solution:

(a) The element stiffness matrices for elements 1 & 2 are

$$K_1 = \frac{E_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{70 \times 10^9 \times 10^{-6} \times 2400}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow K_1 = 10^6 \begin{bmatrix} 1 & 2 \\ 0.56 & -0.56 \\ -0.56 & 0.56 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$K_2 = \frac{E_2 A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{200 \times 10^9 \times 10^{-6} \times 600}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow K_2 = 10^6 \begin{bmatrix} 2 & 2 \\ 0.30 & -0.30 \\ -0.30 & 0.30 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

The global stiffness matrix,  $K = K_1 + K_2$

$$K = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.56 + 0.3 & -0.30 \\ 0 & -0.3 & 0.30 \end{bmatrix}$$

$$\Rightarrow K = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix}$$

The global load vector is,  $F = \begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix}$

Now dofs (or nodes) 1 and 3 are fixed. Since we are using penalty approach, therefore a large number  $C$  is ad has to be added to the first (i.e.  $a_{11}$ ) and third (i.e.  $a_{33}$ ) diagonal elements of  $K$ . Choosing ' $C$ ' as follows.

$$C = [0.86 \times 10^6] \times 10$$

$$C = \max |K_{ij}| \times 10^4 = 0.86 \times 10^6 \times 10^4 \\ = 0.86 \times 10^{10}$$

Thus, the modified stiffness matrix is

$$K = 10^6 \begin{bmatrix} 0.56 + 0.86 \times 10^4 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 + 0.86 \times 10^4 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equation is given by

$$KQ = F \\ \Rightarrow 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \\ 8.1127 \times 10^{-6} \end{bmatrix} \text{ mm}$$

### (b) Elemental Stresses

The element stresses are

$$\sigma_1 = E_1 B q_1 = 70 \times 10^3 \times \frac{1}{300-0} [-1 \ 1] \begin{bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{bmatrix}$$

$$\Rightarrow \sigma_1 = 54.27 \text{ MPa} \quad (\because 1 \text{ MPa} = 10^6 \text{ N/mm}^2)$$

$$\begin{aligned}\sigma_2 &= E_2 B Q_2 = E_2 \times \frac{1}{400} \times [-1 \ 1] \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} \\ &= 200 \times 10^3 \times \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{bmatrix}\end{aligned}$$

$$\Rightarrow \sigma_2 = -116.2809 \text{ MPa}$$

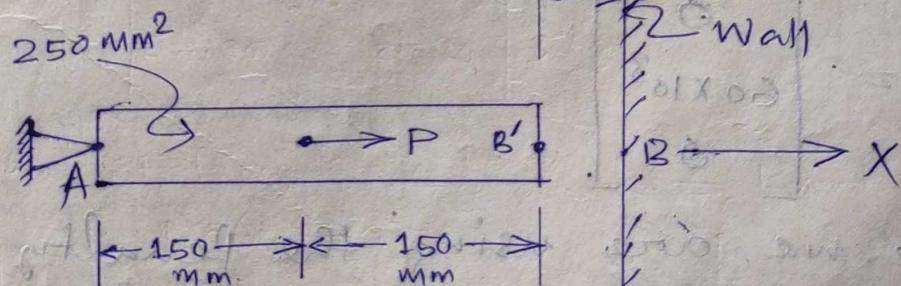
### (c) Reaction Forces:

The reaction forces  $R_1$  and  $R_2$  at support def 1 and 3 can be obtained as,

$$\begin{aligned}R_1 &= -CQ_1 \\ &= -[0.86 \times 10^{10}] \times 15.1432 \times 10^{-6} \\ &= -130.23 \times 10^3 \text{ N} \\ &= -130.23 \text{ KN}\end{aligned}$$

$$\begin{aligned}R_2 &= -CQ_3 \\ &= -[0.86 \times 10^{10}] \times 8.1127 \times 10^{-6} \\ &= -67.77 \times 10^3 \text{ N} \\ &= -67.77 \text{ KN}\end{aligned}$$

Problem-2: In fig. shown below, a load  $P = 60 \times 10^3 \text{ N}$  is applied as shown. Determine the displacement field, stresses and support reactions in the body. Take  $E = 20 \times 10^3 \text{ N/mm}^2$ .



Solution: In this problem, we assume that the wall does not exert. Then, the solution to the problem can be verified to be

$$Q_1 = 1.2 \text{ mm}$$

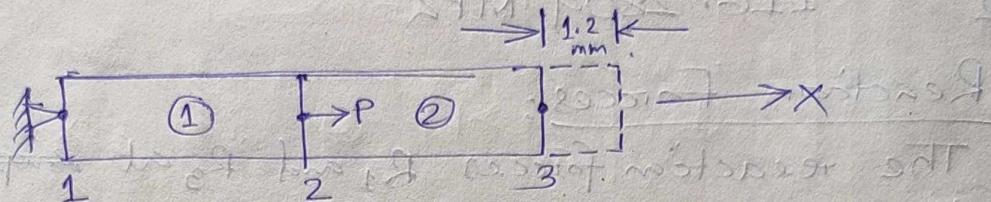
where  $Q_B$  is the displacement of point B.

From it, we see that contact does occur.

Now the problem can be solved by considering the two-element finite element model.

The boundary conditions are  $Q_1 = 0$

$$Q_3 = 1.2 \text{ mm.}$$



The element stiffness matrix for element 1 is

$$K_1 = \frac{EA}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{20 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = \frac{EA}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{20 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global stiffness matrix will be,

$$K = K_1 + K_2$$

$$\Rightarrow K = \frac{20 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The global load vector F is

$$F = \begin{bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{bmatrix}$$

Since we are using the penalty approach, therefore a large number 'C' has to be chosen and added to the first and third diagonal element of K. Choosing 'C' as follows

$$C = \max |K_{ij}| \times 10^4$$

$$\Rightarrow C = \frac{20 \times 10^3 \times 250 \times 2}{150} \times 10^4 = \frac{10^8}{3} \times 20000$$

Thus, the modified stiffness matrix will be

$$K = \frac{10^5}{3} \begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}$$

The finite element equation is given by

$$KQ = F$$

$$\Rightarrow \frac{10^5}{3} \begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 60 \times 10^3 \\ 80 \times 10^7 \end{bmatrix}$$

$\uparrow CQ_3 + F_2 = F_3$

(a)  $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 2.49985 \times 10^{-5} \\ 1.500045 \\ 1.200015 \end{bmatrix}$  mm.

### Elemental Stresses:

The element stresses are

$$\sigma_1 = EBQ = 200 \times 10^3 \times \frac{1}{150} [-1 \ 1] \begin{bmatrix} 2.49985 \times 10^{-5} \\ 1.50045 \end{bmatrix}$$
$$= 199.996 \text{ MPa}$$

$$\sigma_2 = EBQ = 200 \times 10^3 \times \frac{1}{150} [-1 \ 1] \begin{bmatrix} 1.50045 \\ 1.20015 \end{bmatrix}$$
$$= -40.004 \text{ MPa}$$

### Reaction Forces:

The reaction forces,  $R_1 = -CQ_1 = -\frac{10^5}{3} \times 20000 \times \frac{1}{2.49985 \times 10^{-5}}$

$$= -49.999 \times 10^3 \text{ N.}$$

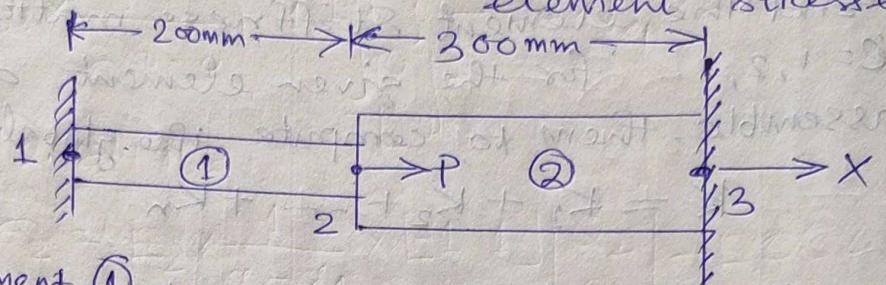
and  $R_3 = -CQ_3 = -\frac{10^5}{3} \times 20000 \times (1.20015 - 1.2)$

$$= -10.001 \times 10^3 \text{ N. kg}$$

Problem - 1:- An axial load  $P = 300 \times 10^3 \text{ N}$  is applied at  $20^\circ\text{C}$  to the rod as shown in fig below. The temp. is then raised to  $60^\circ\text{C}$ .

(a) Determine the global stiffness matrix  $K$  and the global load vector  $F$ .

(b) Determine the nodal displacements and element stresses.



Element ①

Aluminicem

$$E_1 = 70 \times 10^9 \text{ N/m}^2$$

$$A_1 = 900 \text{ mm}^2$$

$$\alpha_1 = 23 \times 10^{-6} \text{ per } ^\circ\text{C}$$

Element ②

Steel

$$E_2 = 200 \times 10^9 \text{ N/m}^2$$

$$A_2 = 1200 \text{ mm}^2$$

$$\alpha_2 = 11.7 \times 10^{-6} \text{ per } ^\circ\text{C}$$

Solution:

(a) The element stiffness matrices are

$$K_1 = \frac{E_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{70 \times 10^9 \times 10^{-6} \times 900}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ N/mm}$$

$$= 10^3 \begin{bmatrix} -315 & -315 \\ -315 & 315 \end{bmatrix} \text{ N/mm}$$

$$K_2 = \frac{E_2 A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{200 \times 10^9 \times 10^{-6} \times 1200}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix} \text{ N/mm}$$

The global stiffness matrix,  $K = K_1 + K_2$

$$K = 10^3 \begin{bmatrix} 315 & -315 & 0 \\ -315 & 315+800 & -800 \\ 0 & -800 & 800 \end{bmatrix} = \begin{bmatrix} 315 & -315 & 0 \\ -315 & 1125 & -800 \\ 0 & -800 & 800 \end{bmatrix} \times 10^3 \text{ N/mm}$$

To find out the global load vector  $F$ , we should first calculate the element temp. load vectors as follows.

$$\Omega_e = \frac{E_e A_e l_e \alpha \Delta T}{x_2 - x_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\Omega_1 = \frac{E_1 A_1 l_1 \alpha \Delta T}{x_2 - x_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \frac{70 \times 10^9 \times 10^{-6} \times 900 \times 200 \times 23 \times 10^{-6} \times (60-20)}{300-0} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \Omega_1 = 10^3 \begin{Bmatrix} -57.96 \\ 57.96 \end{Bmatrix} \text{ N}$$

$$\text{Similarly, } \Omega_2 = \frac{E_2 A_2 l_2 \alpha \Delta T}{x_2 - x_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \frac{200 \times 10^9 \times 10^{-6} \times 1200 \times 300 \times 11.7 \times 10^{-6} \times (60-20)}{300-0} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \Omega_2 = 10^3 \begin{Bmatrix} -112.32 \\ 112.32 \end{Bmatrix} \text{ N}$$

Now by assembling  $\Omega_1$ ,  $\Omega_2$  and the point load 'P' at node 2, give global stiffness matrix vector,  $F = 10^3 \begin{Bmatrix} -57.96 \\ 57.96 + 112.32 + 300 \\ 112.32 \end{Bmatrix}$

$$\Rightarrow F = 10^3 \begin{Bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{Bmatrix} \text{ N.}$$

- (b) The elimination approach can also be used to solve for displacements  $Q$ . Since dofs 1 and 3 are fixed, the first and third rows of and columns of  $K$ , together with the first and third components of  $F$ , are deleted. Thus,

$$10^3 \times 1115 \times Q_2 = 245.64 \times 10^3$$

$$\Rightarrow Q_2 = \frac{245.64 \times 10^3}{1115} = 0.220 \text{ mm}$$

$$\text{Thus, } Q = [0 \quad 0.220 \quad 0]^T \text{ mm.}$$

## Element Stress Calculation:

In evaluating the element stresses, we have use

$$\sigma_e = EBq - E\alpha \Delta T = E \times \frac{1}{x_2 - x_1} [-1 \ 1] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} - E\alpha \Delta T$$

$$\sigma_1 = E_1 \times \frac{1}{x_2 - x_1} [-1 \ 1] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} - E_1 \alpha \Delta T$$

$$= 70 \times 10^3 \times \frac{1}{200-0} [-1 \ 1] \begin{bmatrix} 0 \\ 0.22 \end{bmatrix} - 70 \times 10^3 \times 23 \times 10^{-6} \times (60-20) \\ = 12.6 \text{ MPa}$$

$$\text{Similarly, } \sigma_2 = E_2 \times \frac{1}{x_2 - x_1} \times [-1 \ 1] \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} - E_2 \alpha \Delta T$$

$$= 200 \times 10^3 \times \frac{1}{300-0} \times [-1 \ 1] \begin{bmatrix} 0.22 \\ 0 \end{bmatrix} - 200 \times 10^3 \times 10^{-6} \times 11.7 \times 10^{-6} \times 40 \\ = -240.27 \text{ MPa}$$

Support reactions  $R_1$  and  $R_3$

The element support reactions can be found as

$$R_1 = 10^3 \begin{bmatrix} 315 & -315 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.22 \\ 0 \end{bmatrix} - (-57.96 \times 10^3) \\ = -11340 \text{ N} \\ = -11.34 \text{ KN} \quad \underline{\text{Ans}}$$

$$R_3 = 10^3 \begin{bmatrix} 0 & -800 & 800 \end{bmatrix} \begin{bmatrix} 0 \\ 0.22 \\ 0 \end{bmatrix} - 122.32 \times 10^3 \\ = -298320 \text{ N} \\ = -298.32 \text{ KN} \quad \underline{\text{Ans}}$$

Ans