Mechanical Vibrations
[MV]

Module -III

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Objective-Type
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Set-1 : **OBJECTIVE-TYPE QUESTIONS**

(1) All the moving parts of the system oscillating in the same frequency and phase are known as

(a) principle coordinates

(b) first principal mode of vibration

(c) generalised coordinates

(d) principal mode of vibration

(2) Static coupling occurs due to

(a) static displacements and dynamic inertia forces

(b) static displacements

(c) dynamic inertia forces

(d) all the above statements are true

(3) Dynamic vibration absorber means

(a) it is possible to make the amplitude of vibration of first mass to become zero

(b) it is possible to make the amplitude of vibration of second mass to become zero

(c) it is possible to make the amplitude of vibration of first mass to become maximum

(d) it is possible to make the amplitude of vibration of mass become zero

(4) In case of a two-degree-freedom system, masses will vibrate in two different modes called as

(a) principal-mode vibration

(b) normal-mode vibration

(c) first-mode vibration

(d) none of the above

(5) Centrifugal vibration absorbers are very effective when

(a) at only one frequency of design

(b) either the speed changes or the speed fluctuates

(c) only the speed fluctuates

(d) all of the above cases

(6) In case of a semidefinite system, natural frequency becomes

(a) one of their natural frequencies is equal to zero

(b) their natural frequency becomes maximum

(c) both natural frequencies become zero

(d) both of their natural frequencies are equal

(7) The principal modes or normal modes of vibration for systems

(a) having two or more degrees of freedom are orthogonal

(b) are an important property while finding the mode shape

(c) are an important property while finding the mode shapes and nodes

(d) all of the above cases

(8) In a two-rotor system as shown in Fig. p.6.8.1, if $I_1 > I_2$, a node of vibration lies in

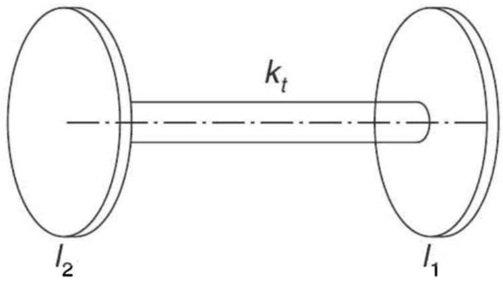


Fig. p.6.8.1

(a) between I_1 and I_2 but near to I_1

(b) between I_1 and I_2 but near to I_2

(c) exactly center between the rotor I_1 and I_2

(d) near I_2 but outside

(9) In a two-rotor system as shown in Fig. p.6.8.1, the frequency equation is given by

(a) $\omega_1 = \sqrt{\dfrac{I_1 I_2}{k_t\,(I_1 + I_2)}}$ rad/s

(b) $\omega_2 = \sqrt{\dfrac{k_t\,(I_1 + I_2)}{I_1 I_2}}$ rad/s

(b) $\omega_1 = \sqrt{\dfrac{k_t(I_1 + I_2)}{I_1 I_2}}$ rad/s

(d) $\omega_2 = \sqrt{\dfrac{I_1 I_2}{k_t\,(I_1 + I_2)}}$ rad/s

(10) Lagrange's method is very suitable for

(a) the presence of damping force present in a system

(b) the presence of force present in a system

(c) the presence of force in function and damping forces present in a system

(d) all of the above cases

Answers

(1) d	(2) b	(3) d	(4) a	(5) b	(6) a
(7) a	(8) d	(9) b	(10) c		

Set-2 : OBJECTIVE-TYPE QUESTIONS

(1) A shaft carrying three rotors will have
   (a) no node (b) three nodes
   (c) two nodes (d) one node

(2) The natural frequencies and mode shapes can be determined easily and quickly in case of multi-degree-freedom systems
   (a) by solving the number of equations easily
   (b) easily and quickly with the help of computers
   (c) by applying Newton's second law of motion only
   (d) none of the above cases

(3) Influence coefficient can be determine by using
   (a) Maxwell's reciprocal theorem
   (b) matrix iteration method
   (c) orthogonality principle
   (d) Rayleigh–Ritz method

(4) Matrix iteration methods are used to determine
   (a) analysis of problems in natural frequencies only
   (b) analysis of problems in structures, vibrations, fluid dynamics and design
   (c) large number of mathematical equations
   (d) all of the above cases

(5) Matrix iteration method is used in
   (a) determining amplitudes of second and third modes of vibration
   (b) determining the principle modes or normal modes of vibration
   (c) an iterative procedure to determine the principal modes of the system and its natural frequencies
   (d) an important property while finding the fundamental natural frequencies

(6) In a four-degree-freedom system, the eigenvalue will be
   (a) two eigenvalues
   (b) three eigenvalues
   (c) four eigenvalues
   (d) zero eigenvalue

(7) In eigenvalue problems
   (a) the eigenvector will represent the mode shape
   (b) the eigenvector will represent the natural frequency
   (c) the eigenvector will represent the mode shape as well as natural frequency
   (d) all of the above cases

(8) For a 3-degree freedom system, orthogonality principle can be written as
   (a) $m_1A_1A_2 + m_2B_1B_2 = 0$
   (b) $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$
   (c) $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$
   $m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$
   $m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$
   (d) None of the above cases

Answers

(1) c (2) b (3) a (4) b (5) c (6) c
(7) a (8) c

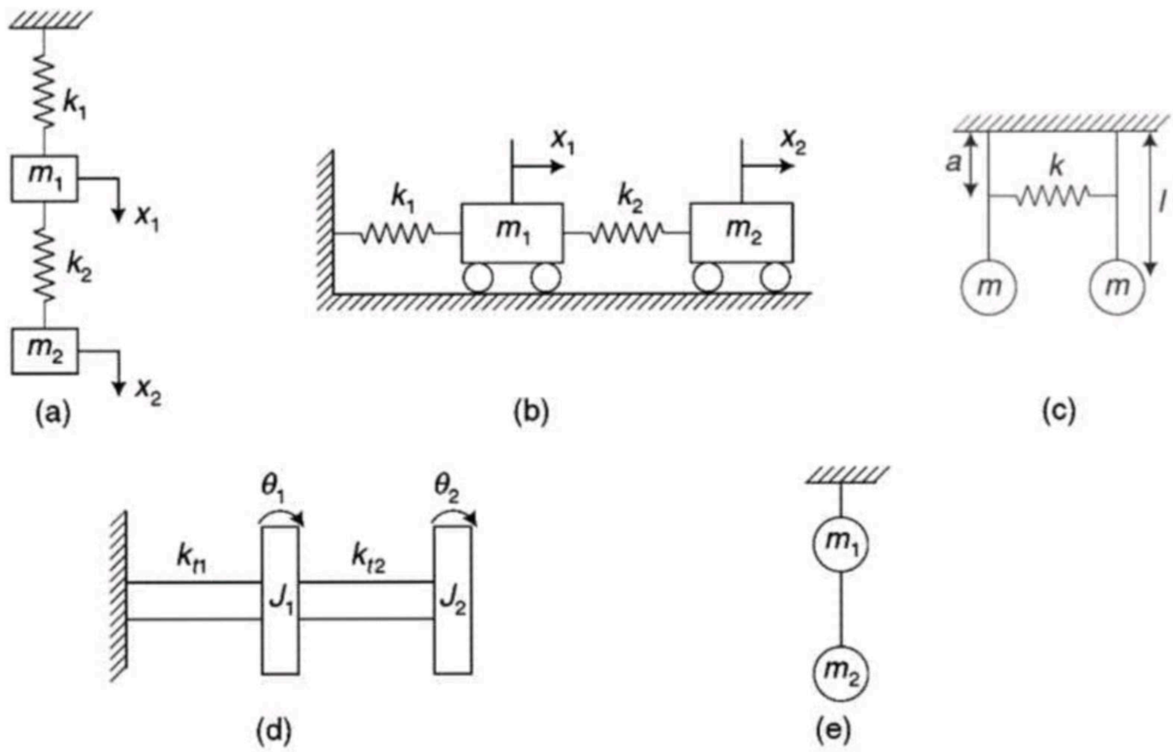
SHORT-TYPE

Questions & Answers

Bijan Kumar Giri

Q. What do you mean by Two degree of freedom system .

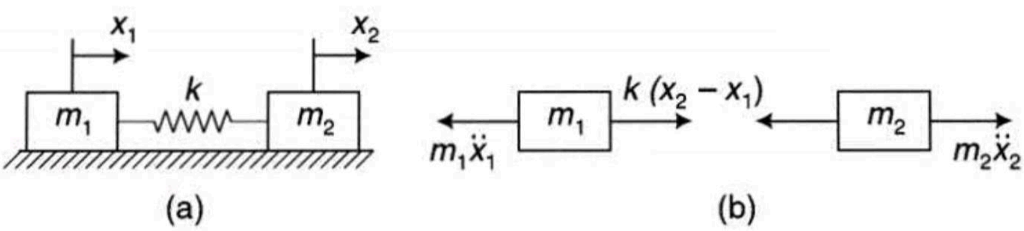
Systems that require two independent coordinates to specify the system configuration at any instant are called **'two-degree-freedom systems'**. In such a system there are two masses which have two equations of motion, treated as coupled differential equations. Each mass will have its own natural frequency. Sometimes nonharmonic motion of the masses makes the system more complicated for solving problems.



Two-degree-freedom systems

Q. Define semi-definite system .

This is defined as a system where one natural frequency is equal to zero. This is also known as a degenerate system. Consider the system to represent two masses 'm_1' and 'm_2' and with a coupling spring 'k' as shown in Fig. 6.2(a).



Semidefinite system

Q. What is Orthagonality Principle.

The principal modes or normal modes of vibration for systems having two or more degrees of freedom are orthogonal. This is known as orthogonality principle.

This is an important property while finding the natural frequencies.

For a two-degree-freedom system, orthogonality principle can be written as

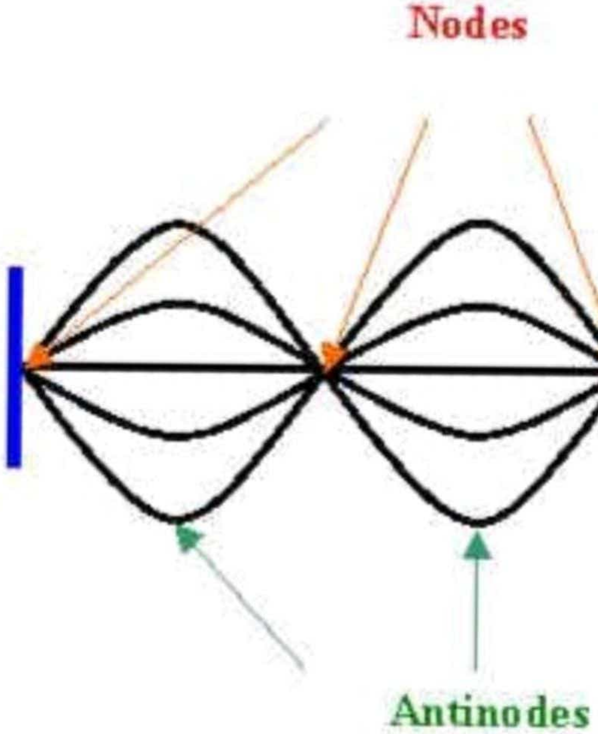
$$m_1A_1A_2 + m_2B_1B_2 = 0,$$

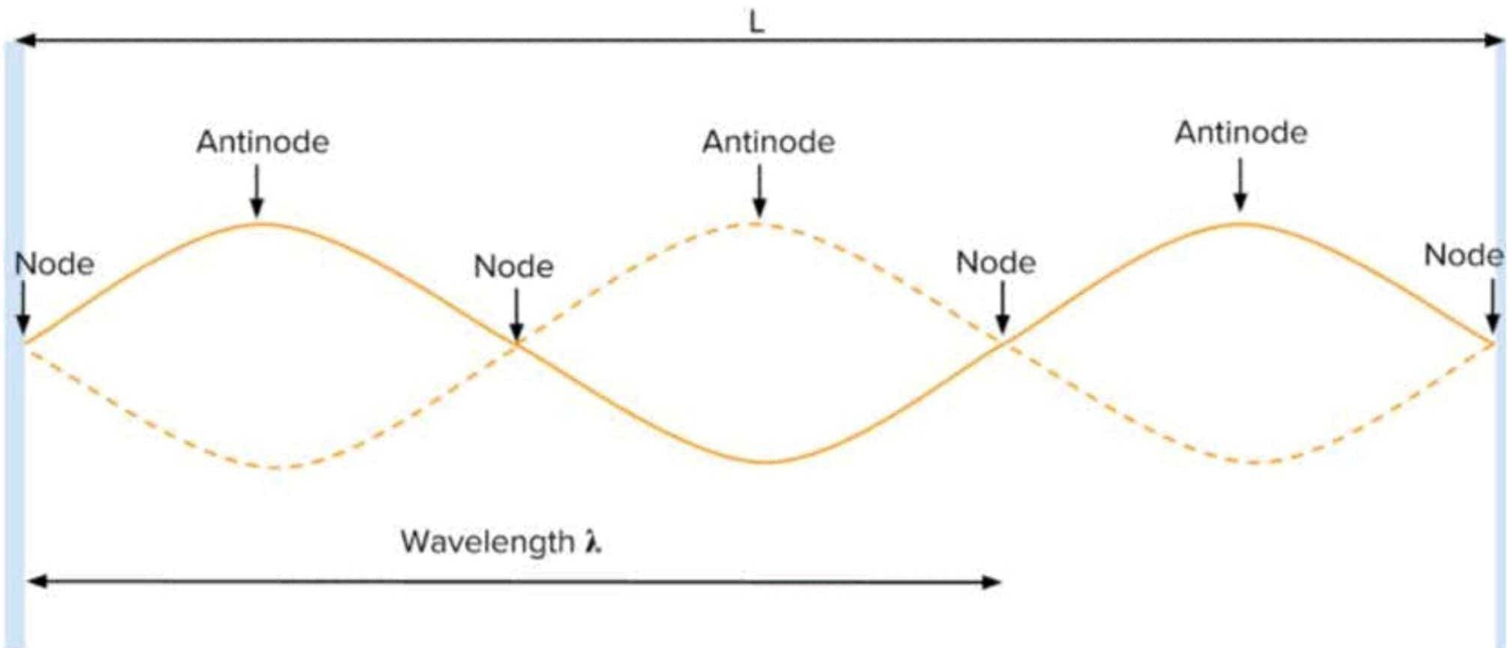
where A_1, B_1 and A_2, B_2 are the amplitudes of first and second modes of vibration.

Q. Define Node and Antinode in Vibration.

Ans : There exists nodes and anti nodes in the vibration.

The point which vibrate with zero amplitude i.e. The stationary point is called node. The point of maximum amplitude vibration is called antinode. If u want to see them fix a string at both ends and pluck it.





LONG -TYPE

Questions & Answers

Bijan Kumar Giri

EXAMPLE 6.1

Determine the natural frequencies for the following system as shown in Fig. p-6.1(a) and determine the ratio of amplitudes and locates the nodes for each mode of vibration and draw the mode shapes. Given, $m_1 = m$, $k_1 = 2k$, $m_2 = 2m$, $k_2 = k$.

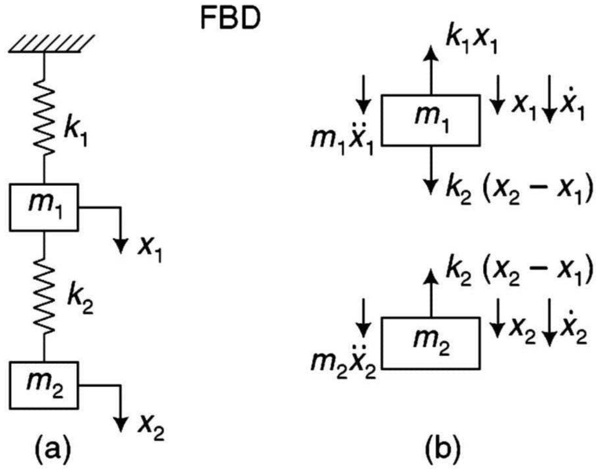


Fig. p-6.1 *Two-degree linear spring-mass system*

Solution Now at any instant, give displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.1(a). The FBD is as shown in Fig. p-6.1(b).

Applying Newton's second law of motion to mass 'm_1', assuming that $x_2 > x_1$

$$\Sigma F = m\,a$$

$$\therefore \quad k_2(x_2 - x_1) - k_1x_1 = m_1\ddot{x}_1$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0$$

$$\therefore \quad m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

But the given values of $m_1 = m$, $k_1 = 2k$, $k_2 = k$.

$$\therefore \quad m\ddot{x}_1 + (2k + k)x_1 - kx_2 = 0,\; m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad ...6.1$$

This is the differential equation of motion of the mass 'm_1'.

Again apply Newton's second law of motion to the mass 'm_2'.

$$\Sigma F = m\,a$$

$$\therefore \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

But the given values of $k_1 = 2k$, $m_2 = 2m$, $k_2 = k$

$$\therefore \quad 2m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad ...\ 6.2$$

This is the differential equation of motion of the mass 'm_2'.

Assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$
$$\dot{x}_1 = \omega A \cos \omega t \qquad \dot{x}_2 = \omega B \cos \omega t$$
$$\ddot{x}_1 = -\omega^2 A \sin \omega t \quad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Using the values of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.1, we get

$$m(-A\omega^2 \sin \omega t) + 3kA \sin \omega t - kB \sin \omega t = 0$$
$$-m\omega^2 A \sin \omega t + 3\,kA \sin \omega t = kB \sin \omega t$$
$$A \sin \omega t\,(3k - m\omega^2) = kB \sin \omega t,\; A\,(3k - m\omega^2) = kB$$

The amplitude ratio $\therefore \dfrac{A}{B} = \dfrac{k}{3k - m\omega^2}$ .. 6.3

Again using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.2, we get

$$2m(-\omega^2 B \sin \omega t) + k(B \sin \omega t) - k(A \sin \omega t) = 0$$
$$-2m\omega^2 B \sin \omega t + kB \sin \omega t = kA \sin \omega t$$
$$B \sin \omega t\,(k - 2m\omega^2) = kA \sin \omega t,\; B(k - 2m\omega^2) = kA$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k - 2m\omega^2}{k}$ ...6.4

From equations 6.3 and 6.4,

$$\frac{k}{3k - m\omega^2} = \frac{k - m\omega^2}{k}$$
$$(3k - m\omega^2)\,(k - 2m\omega^2) = k^2$$
$$3k^2 - 2m\omega^2 \times 3k - m\omega^2 k + 2m^2\omega^4 = k^2,\; 2m^2\omega^4 - 7km\omega^2 + 2k^2 = 0$$
$$\omega^4 - \frac{7k}{2m}\,\omega^2 + \frac{k^2}{m^2} = 0$$

This is a quadratic equation in ω^2, where roots are given by

$$\omega^2 = \frac{+\dfrac{7k}{2m} \pm \sqrt{\left(\dfrac{7k}{2m}\right)^2 - \dfrac{4k^2}{m^2}}}{2},\; \omega^2 = \frac{7k}{4m} \pm \frac{\sqrt{\dfrac{49k^2}{4m^2} - \dfrac{4k^2}{m^2}}}{4}$$

$$\omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{49k^2 - 16k^2}{16m^2}},\; \omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{33k^2}{16m^2}}$$

$$\omega^2 = \frac{7k}{4m} \pm \frac{5.74k}{4m},\; \omega_{1n}^2 = \frac{7k}{4m} - 5.74k/4m,\; \omega_{2n}^2 = \frac{7k}{4m} + \frac{5.74k}{4m}$$

$$\omega_{1n}^2 = 0.315\,\frac{k}{m}\; \omega_{2n}^2 = 3.185\,\frac{k}{m},\; \omega_{1n} = 0.56\sqrt{\frac{k}{m}}\text{ rad/s},\; \omega_{2n} = 1.78\sqrt{\frac{k}{m}}\text{ rad /s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

To draw the mode shapes

(i) First mode shape. From Eq. 6.3

$$\frac{A}{B} = \frac{k}{3k - m\omega^2}$$

At $\quad \omega^2 = \omega_{1n}^2 = 0.315\,\frac{k}{m}, \qquad \frac{A}{B} = \frac{k}{3k - m \times 0.315\,\frac{k}{m}}$

$\therefore \quad \frac{A}{B} = \frac{1}{2.69}$, i.e. at $A = 1$, $B = 2.6$ [Fig. p-6.1(c)]

(ii) Second mode shape. From Eq. 6.4, $\frac{A}{B} = \frac{k - 2m\omega^2}{k}$

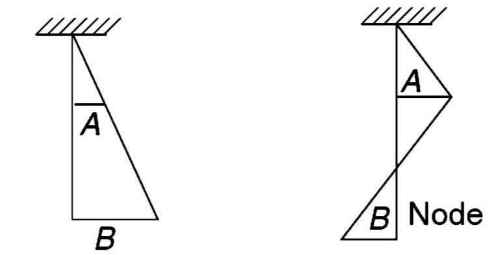
At $\quad \omega^2 = \omega_{2n}^2 = 3.185\,\frac{k}{m}$

At $\quad A = 1$

$$\therefore \quad \frac{A}{B} = \frac{k - 2m \times 3.185\,\frac{k}{m}}{k} = \frac{k - 6.37k}{k}$$

$$\therefore \quad \frac{A}{B} = -5.37$$

$$\therefore \quad A = -5.37\,B \text{ [Fig. p-6.1(d)]}$$



(c) Same phase (d) Out of phase

Fig. p-6.1 *Mode shape*

In the first mode of Fig. p-6.1(c) the full spring moves to the right side of the mean line as it is ‘same phase’.

Whereas in the second mode of Fig. p-6.1(d), the second spring crosses the mean line as it is ‘out of phase’. The crossed point is called **‘node’** point, i.e. there is no displacement at that point.

Note: (i) Node is a point in a vibrating system which doesn’t experience any displacements.

(ii) As number of modes (degree) increases, number of nodes also increases.

EXAMPLE 6.2

Find the natural frequency of the system as shown in Fig. p-6.2(a).

Solution Now at any instant give displacement ‘x_1’ to the mass ‘$2m$’ and ‘x_2’ to the mass ‘m’ to Fig. p-6.2(a). The FBD is as shown in Fig. p-6.2(b).

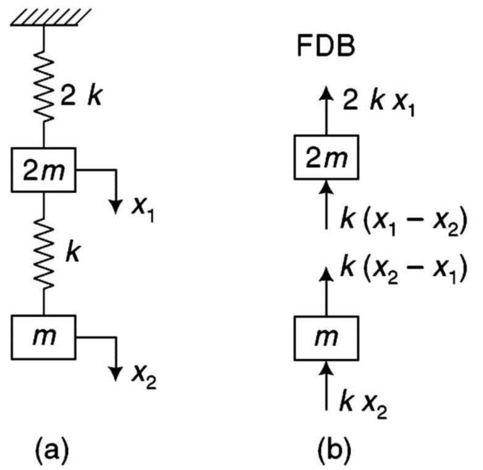


Fig. p-6.2 *Two-degree linear spring-mass system*

Applying Newton's second law of motion to mass '$2m$', assuming that $x_2 > x_1$

$$\Sigma F = m\,a,\ 2m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad \text{...6.5(a)}$$

Newton's second law of motion to mass 'm', $\quad m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad$ ...6.5(b)

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \qquad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Using the values of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.5 and again using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.5(a) and rearranging, we have $3k - 2m\ \omega^2\ A - kB = 0$, $-kA + (k - m\omega^2)$ $B = 0$.

The determents of the coefficient of 'A' and 'B' are

$$\begin{bmatrix} 3k - 2m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} = 0,\ (3k - 2m\omega^2)\,(k - m\omega^2) - k^2 = 0,$$

$$2m^2\omega^4 - 5\ km\ \omega^2 + 2\ k^2 = 0$$

$$m\omega\tau^2 = \frac{5k \pm \sqrt{(5k)^2 - 16k^2}}{4} = \frac{5k \pm 3k}{4} = \frac{k}{2} \text{ or } 2k$$

$$m\omega_{1n}^2 = \frac{k}{2} \text{ or } \omega_{1n}^2 = \frac{k}{2m},\ \omega_{1n} = \sqrt{\frac{k}{2m}} \text{ and mode shape } \left(\frac{B}{A}\right)_1 = \frac{3k - 2m\omega_{1n}^2}{k} = 2$$

$$m\omega_{2n}^2 = 2k,\ \omega_{2n} = \sqrt{\frac{2k}{m}},\ \text{and mode shape } \left(\frac{B}{A}\right)_1 = -1$$

EXAMPLE 6.3

Determine the natural frequency and normal modes for the system as shown in Fig. p-6.3(a). Draw the mode shape and locate the node.

Solution Let us at any instant give a vertical displacement 'x_1' to the first mass 'm' and 'x_2' be the second mass as shown in Fig. p-6.3(a). Then the FBD is as shown in Fig. p-6.3(b). Now apply Newton's second law of motion to 'm' (rectilinear motion).

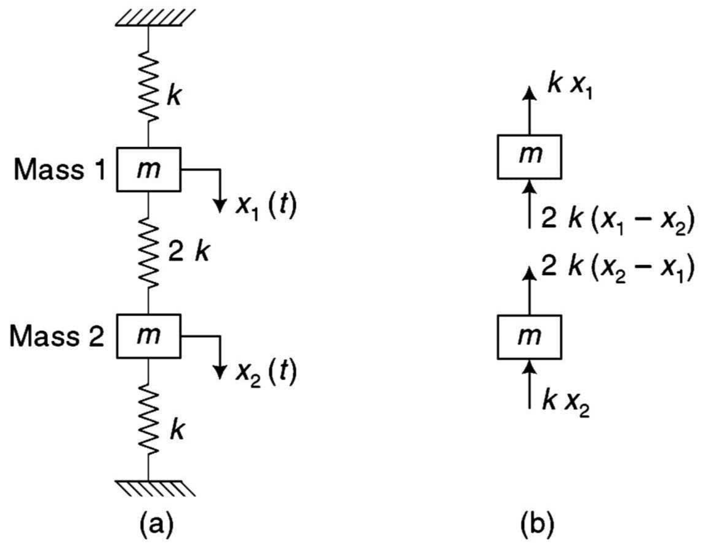


Fig. p-6.3 *Two-degree linear spring-mass system*

From Newton's second law of motion, $\Sigma F = ma$

$$m\ddot{x}_1 = -kx_1 - 2k(x_1 - x_2), \qquad m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2$$

$$m\ddot{x}_1 + kx_1 + 2k(x_1 - x_2) = 0 \qquad \text{...6.6}$$

This is the differential equation of motion for the mass (1)

$$m\ddot{x}_2 + 2k(x_2 - x_1) + kx_2 = 0 \qquad \text{...6.7}$$

This is the differential equation of motion for the mass (2)

Assuming that the motion is periodic and is composed of harmonic motions of varies amplitudes and frequencies, let one of these components be,

$$x_1 = X_1 \sin \omega t \qquad x_2 = X_2 \sin \omega t$$

$$\dot{x}_1 = \omega X_1 \cos \omega t \qquad \dot{x}_2 = \omega X_2 \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin \omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$

Using these values in Eq. 6.6 and Eq. 6.7, we have

$[-m\omega^2 x_1 + k_1 x_1 + 2k(x_1 - x_2)] \sin \omega t = 0,$

$[-m\omega^2 x_2 + 2k(x_2 - x_1)\, kx_2 +] \sin \omega t = 0$

$\sin \omega t \neq 0$

$$(3k - m\omega^2)\, x_1 - 2kx_2 = 0, \qquad (3k - m\omega^2)\, x_2 - 2kx_1 = 0 \qquad \text{...6.8}$$

The determinant of the coefficient $\begin{vmatrix} x_1 & x_2 \\ (3k - m\omega^2) & (-2k) \\ (-2k) & (3k - m\omega^2) \end{vmatrix} = 0,$

$$(3k - m\omega^2)(3k - m\omega^2) - (-2k)(-2k) = 0$$

$$9k^2 - 3km\omega^2 - 3\,km\omega^2 + m^2\omega^4 - 4k^2 = 0$$

$$9k^2 - 6km\omega^2 + m^2\omega^4 - 4k^2 = 0 \qquad m^2\omega^4 - 6km\omega^2 + 5k^2 = 0 \text{ divided by } m^2$$

$$\omega^4 - \left(\frac{6k}{m}\right)\omega^2 + 5\left(\frac{k}{m}\right)^2 = 0, \qquad \omega_{1,2}^2 = \frac{\frac{6k}{m} \pm \sqrt{\left(\frac{6k}{m}\right)^2 - 4 \times 5\left(\frac{k}{m}\right)^2}}{2}$$

$$\omega_{1,2}^2 = \frac{3k}{m} \pm \sqrt{\left(\frac{3k}{m}\right)^2 - 5\left(\frac{k}{m}\right)^2}, \quad \omega_{1,2}^2 = \frac{3k}{m} \pm \sqrt{9\left(\frac{k}{m}\right)^2 - 5\left(\frac{k}{m}\right)^2} = \frac{3k}{m} \pm 2\,\frac{k}{m}$$

$$\omega_1^2 = \frac{3k}{m} - 2\,\frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m} + 2\,\frac{k}{m}, \quad \omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{5k}{m}$$

$$\omega_1 = \sqrt{\frac{k}{m}}\ \text{rad/s}, \quad \omega_2 = \sqrt{\frac{5k}{m}}\ \text{rad/s}$$

From Eq. 6.8, $\left(\frac{x_1}{x_2}\right) = \dfrac{-2k}{(3k - m\omega_1^2)} = \dfrac{2k}{\left(3k - m \times \frac{k}{m}\right)} = \dfrac{2k}{2x} = +1$

$$\left(\frac{x_1}{x_2}\right)_2 = \frac{\left(3k - m\omega_2^2\right)}{2k} = \frac{\left(3k - m \times \frac{5k}{m}\right)}{2k} = \frac{-2k}{2x} = -1 \quad \frac{x_{11}}{x_{21}} = +1 \quad \frac{x_{12}}{x_{22}} = -1$$

To draw the mode shapes

Take $x_{11} = 1, \quad x_{21} = 1, \quad x_{12} = -1, \quad x_{22} = -1$

The first-mode and second-mode shapes are as shown in Fig. p-6.3(c) and Fig. p-6.3(d).

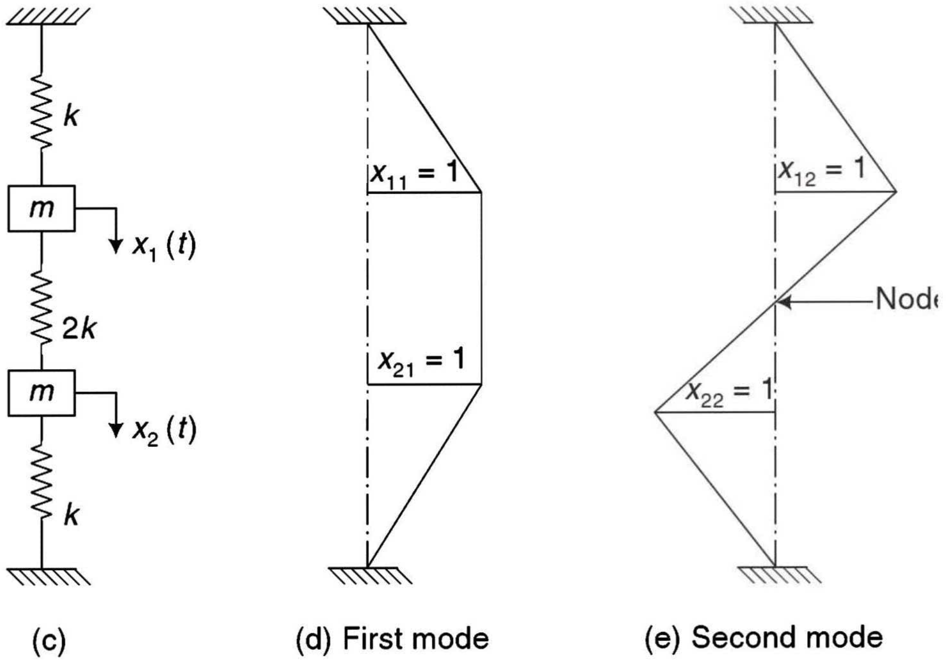


Fig. p-6.3 *Mode shape*

EXAMPLE 6.4

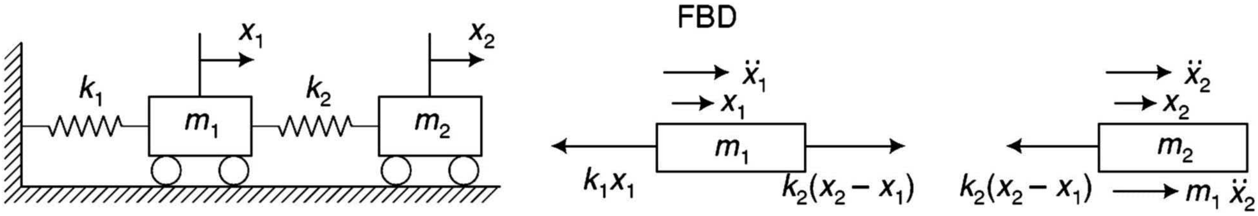
Determine the equation of motion and the natural frequencies of the two-degree-freedom system shown in Fig. p-6.4(a). Determine the displacements 'x_1' and 'x_2' in terms of natural frequencies.

Solution Now at any instant give linear displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.4(a). The FBD is as shown in Fig. p-6.4(b).

Now applying Newton's second law of motion to mass 'm_1', assuming that $x_2 > x_1$

$$\Sigma F = m\ddot{x} \quad \rightarrow \text{Take +ve}, \quad \leftarrow \text{Take –ve}$$

$$\therefore \quad -k_1x_1 + k_2\,(x_2 - x_1) = m_1\,\ddot{x},\ m_1\,\ddot{x}_1 + k_1x_1 - k_2\,(x_2 - x_1) = 0$$



(a) (b)

Fig. p-6.4 *Two-degree linear spring-mass system*

$$m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0,\ m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \qquad ...6.9$$

This is a differential equation of motion of mass 'm_1'.

Applying Newton's second law of motion to mass 'm_2'$\Sigma F = m\ddot{x}$,

$$\therefore \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2,\ m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0,\ m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \qquad ...6.10$$

This is a differential equation of motion of mass 'm_2'.

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A\sin\omega t \qquad x_2 = B\sin\omega t$$

$$\ddot{x}_1 = -A\omega^2\sin\omega t \qquad \ddot{x}_2 = -B\omega^2\sin\omega t$$

Using the value of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.9,

$$m_1(-A\omega^2\sin\omega t) + (k_1 + k_2)A\sin\omega t - k_2B\sin\omega t = 0$$

$$-m_1\omega^2A + (k_1 + k_2)A - k_2B = 0,\ A[(k_1 + k_2) - m_1\omega^2] = k_2B$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k_2}{(k_1 + k_2) - m_1\omega^2}$ ...6.11

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.10,

$$m_2(-\omega^2B\sin\omega t) + k_2(B\sin\omega t) - k_2(A\sin\omega t) = 0$$

$$-m_2\omega^2B + k_2B - k_2A = 0, \quad B(k_2 - m_2\omega^2) = k_2A$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k_2 - m_2\omega^2}{k_2}$ ...6.12

From equations 6.11 and 6.12,

$$\frac{k_2}{(k_1 + k_2) - m_1\omega^2} = \frac{k_2 - m_2\omega^2}{k_2},\ (k_2 - m_2\omega^2)[(k_1 + k_2) - m_1\omega^2] = k_2^2$$

$$k_2(k_1 + k_2) - m_1k_2\omega^2 - m_2(k_1 + k_2)\omega^2 + m_1m_2\omega^4 - k_2^2 = 0$$

$$m_1m_2\omega^4 - m_1k_2\omega^2 - m_2k_1\omega^2 - m_2k_2\omega^2 + k_1k_2 + k_2^2 - k_2^2 = 0$$

$$\omega^4 - \frac{k_2}{m_2}\omega^2 - \frac{k_1}{m_1}\omega^2 - \frac{k_2}{m_1}\omega^2 + \frac{k_1k_2}{m_1m_2} = 0$$

$$\omega^4 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]\omega^2 - \frac{k_1k_2}{m_1m_2} = 0$$

This is a quadratic equation in ω^2, whose roots are given by

$$\omega^2 = \frac{\left[\dfrac{k_1+k_2}{m_1}+\dfrac{k_2}{m_2}\right] \pm \sqrt{\left(\dfrac{k_1+k_2}{m_1}+\dfrac{k_2}{m_2}\right)^2 - 4\dfrac{k_1k_2}{m_1m_2}}}{2}$$

$$\therefore \qquad \omega^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] \pm \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}}$$

$$\therefore \qquad \omega_{1n}^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] - \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}},$$

$$\therefore \qquad \omega_{2n}^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] + \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}},$$

Hence the general solutions x_1 and x_2 are given by,

$$x_1 = A_1 \sin \omega_{1n}t + A_2 \sin \omega_{2n}t, \quad x_2 = B_1 \sin \omega_{1n}t + B_2 \sin \omega_{2n}t$$

where A_1, A_2 and B_1, B_2 are constants and are evaluated by four initial conditions:

$$x_1(0), \quad \dot{x}_1(0), \quad x_2(0), \quad \dot{x}_2(0).$$

EXAMPLE 6.5

Determine the natural frequency and the amplitude ratios for the system as shown in Fig. p-6.5(a). If the mass 'm_1' is displaced 1 m from its static equilibrium position and released, determine the resulting displacements 'x_1' and 'x_2'. Given $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$.

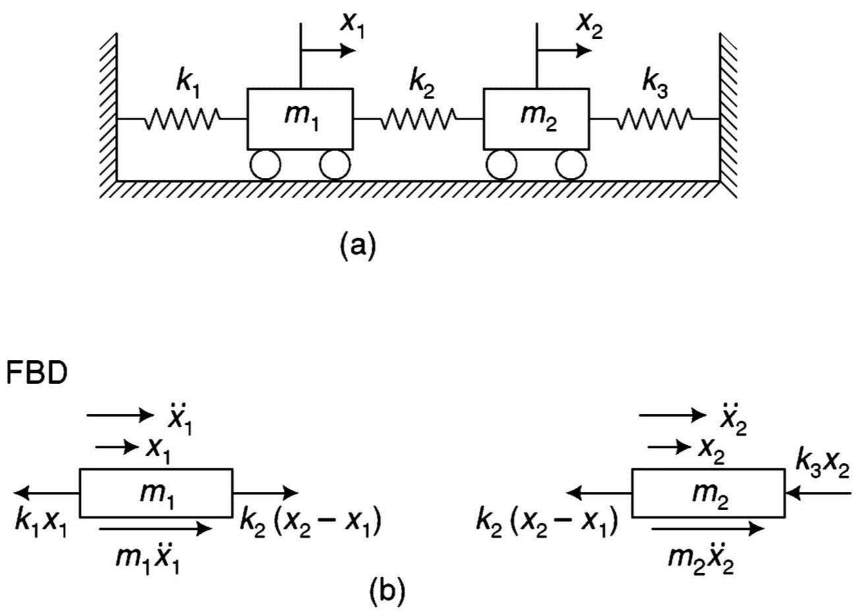


Fig. p-6.5 *Two-degree linear spring-mass system*

Solution Now at any instant give linear displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.5(a). The FBD is as shown in Fig. p-6.5(b).

Now applying Newton's second law of motion to mass 'm_1' assuming that $x_2 > x_1$,

$$k_2(x_2 - x_1) - k_1x_1 = m_1\ddot{x}_1,\ m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0$$

$$m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = 0$$

Given $$m_1 = m,\ k_1 = k_2 = k$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \qquad \text{...6.13}$$

This is the differential equation of motion for the mass 'm_1'.

Again applying Newton's second law of motion to the mass 'm_2',

$$-k_2(x_2 - x_1) - k_3x_2 = m_2\ddot{x}_2,\quad m_2\ddot{x}_2 + k_3x_2 + k_2x_2 - k_2x_1 = 0$$

$$m_2\ddot{x}_2 + (k_3 + k_2)\,x_2 - k_2\,x_1 = 0$$

Given $$m_2 = m,\quad k_2 = k_3 = k$$

$$m\ddot{x}_2 + 2k\,x_2 - k\,x_1 = 0 \qquad \text{...6.14}$$

This is the differential equation of motion for the mass 'm_2'.

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A\cos\omega t,\quad x_2 = B\cos\omega t,\quad \ddot{x}_1 = -A\,\omega^2\cos\omega t,\quad \ddot{x}_2 = -B\,\omega^2\cos\omega t$$

Using the values of $x_1, x_2, \ddot{x}_1$ in Eq. 6.13,

$$m(-A\,\omega^2\cos\omega t) + 2kA\cos\omega t - kB\cos\omega t = 0$$

$$A(2k - m\omega^2) = Bk$$

The amplitude ratio $$\frac{A}{B} = \frac{k}{2k - m\omega^2} \qquad \text{...6.15}$$

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.14,

$$m(-B\cos\omega t)\,\omega^2 + 2kB\cos\omega t - kA\cos\omega t = 0,\ B(2k - m\omega^2) = kA$$

The amplitude ratio $$\frac{A}{B} = \frac{2k - m\omega^2}{k} \qquad \text{...6.16}$$

From equations 6.15 and 6.16,

$$\frac{k}{2k - m\omega^2} = \frac{2k - m\omega^2}{k},\qquad (2k - m\omega^2)^2 = k^2$$

$$4k^2 - 4mk\omega^2 + m^2\omega^4 = k^2,\ m^2\omega^4 - 4mk\omega^2 + 3k^2 = 0,\ \omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

This is the quadratic equation in ω^2, where $$\omega^2 = \frac{\frac{4k}{m} \pm \sqrt{\left(\frac{4k}{m}\right)^2 - \frac{4\times 3k^2}{m^2}}}{2}$$

$$\omega^2 = \frac{2k}{m} \pm \sqrt{\frac{4k^2}{m^2} - \frac{3k^2}{m^2}} = \frac{2k}{m} \pm \frac{k}{m}\sqrt{4-3}$$

$$\omega^2 = \frac{2k}{m} \pm \frac{k}{m}$$

$$\therefore \qquad \omega_{1n}^2 = \frac{k}{m}$$

$$\omega_{2n}^2 = \frac{3k}{m}$$

$$\therefore \qquad \omega_{1n} = \sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{2n} = 1.73\sqrt{\frac{k}{m}} \text{ rad/s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

This resulting displacements 'x_1' and 'x_2' are given by

$$x_1 = A_1 \cos \omega_{1n}t + A_2 \cos \omega_{2n}t \qquad \text{...6.17}$$

$$x_2 = B_1 \cos \omega_{1n}t + B_2 \cos \omega_{2n}t \qquad \text{...6.18}$$

Given $\qquad x_1 = 1 \text{ m at } t = 0$

$$\therefore \qquad \dot{x}_1 = 0 \text{ at } t = 0$$

$$x_2 = 0 \text{ at } t = 0$$

$$\dot{x}_2 = 0 \text{ at } t = 0$$

Differentiating equations 6.17 and 6.18 with respect to time 't',

$$\dot{x}_1 - A_1\,\omega_{1n} \sin \omega_{1n}t + (-A_2\,\omega_{2n} \sin \omega_{2n}t) \qquad \text{...6.19}$$

$$\dot{x}_2 = -B_1\,\omega_{1n} \sin \omega_{1n}t + (-B_2\,\omega_{2n} \sin \omega_{2n}t) \qquad \text{...6.20}$$

From Eq. 6.15, $\dfrac{A}{B} = \dfrac{k}{2k - m\omega^2}$, $\quad$ at $\omega^2 = \omega_{1n}^2 = \dfrac{k}{m}$

$$\frac{A_1}{B_1} = \frac{k}{2k - m\dfrac{k}{m}}, \quad \frac{A_1}{B_1} = 1 \text{ or } A_1 = B_1$$

At $\qquad \omega^2 = \omega_{2n}^2 = \dfrac{3k}{m}, \quad \dfrac{A_2}{B_2} = \dfrac{k}{2k - 3k} = -1$

$$\therefore \qquad A_2 = -B_2$$

Using the initial conditions in equations 6.17, 6.18, 6.19 and 6.20,

$$A_1 + A_2 = 1, \quad B_1 + B_2 = 0, \quad A_1 - A_2 = 0, \quad 2A_1 = 1$$

$$\therefore \qquad A_1 = \frac{1}{2}, \quad B_1 = \frac{1}{2}, \quad A_2 = +\frac{1}{2}, \quad B_2 = -\frac{1}{2}$$

Thus, the motions of the masses are given by

$$X_1 = \frac{1}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{1}{2}\cos\left(1.73\sqrt{\frac{k}{m}}t\right), X_2 = \frac{1}{2}\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{1}{2}\cos\left(1.73\sqrt{\frac{k}{m}}t\right).$$

EXAMPLE 6.6

Find the natural frequency of oscillation of the double pendulum as shown Fig. p-6.6(a) where $m_1 = m_2 = m$, and $l_1 = l_2 = l$. Draw the mode shapes and locate the nodes for each mode of vibration.

Solution Now at any instant give an angular displacement to the bobs in Fig. p-6.6(a) in the horizontal position.

Let 'θ_1' and 'θ_2' be the angular displacements of masses 'm_1' and 'm_2' respectively from vertical equilibrium positions 'x_1' and 'x_2'. Then the FBD is as shown in Fig. p-6.6(b).

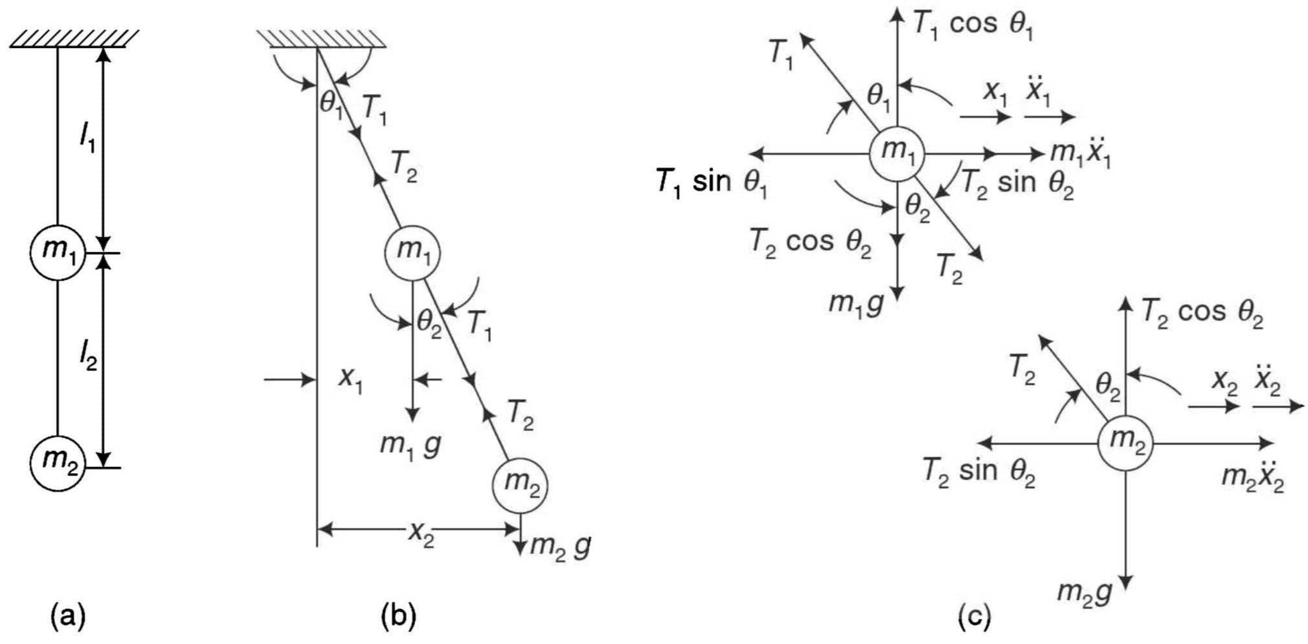


Fig. p-6.6 *Double pendulum*

Now applying Newton's second law of motion to the mass 'm_1' (considering only horizontal forces),

$$-T_1 \sin \theta_1 + T_2 \sin \theta_2 = m_1\ddot{x}_1$$

$\therefore$
$$m_1\ddot{x}_1 + T_1 \sin \theta_1 - T_2 \sin \theta_2 = 0$$

But considering masses 'm_2' and 'm_1',

$$\Sigma v = 0$$

$\therefore$
$$T_2 \cos \theta_2 = m_2 g, \quad T_1 \cos \theta_1 = T_2 \cos \theta_2 + m_1 g$$

For small angles of θ_1 and θ_2, $\cos \theta_1 \approx 1$ and $\cos \theta_2 \approx 1$

$\therefore$
$$T_2 = m_2 g, \quad T_1 = T_2 + m_1 g, T_1 = (m_1 + m_2)g$$

For the geometry of Fig. p-6.6(b),

$$\sin \theta_1 = \frac{x_1}{l_1} \text{ and } \sin \theta_2 = \frac{x_2 - x_1}{l_2}$$

$$m_1\ddot{x}_1 + (m_1 + m_2)g\frac{x_1}{l_1} - \frac{mg}{l}x_2 + \frac{mg}{l}x_1 = 0$$

$$m_1\ddot{x}_1 + \frac{3mg}{l_1}x_1 - \frac{mg}{l}x_2 = 0 \qquad \text{...6.21}$$

Applying Newton's second law of motion to the mass m_2 (considering only horizontal force),

$$-T_2 \sin \theta_2 = m_2 \ddot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + m_2 g = \frac{x_2 - x_1}{l_2} = 0, \; m\ddot{x}_2 + \frac{mg}{l} x_2 - \frac{mg}{l} x_1 \qquad \text{...6.22}$$

Assuming that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\ddot{x}_1 = -A\omega^2 \sin \omega t \qquad \ddot{x}_2 = -B\omega^2 \sin \omega t$$

Using these values x_1, x_2, $\ddot{x}_1$ in Eq. 6.21,

$$m\,(-A\omega^2) + \frac{3mg}{l} A - \frac{mg}{l} B = 0, \quad A\left(\frac{3mg}{l} - m\omega^2\right) = \frac{mg}{l} B$$

$$\frac{A}{B} = \left(\frac{mg}{3mg - ml\omega^2}\right) \qquad \text{...6.23}$$

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.22,

$$m\,(-B\omega^2) + \frac{mg}{l} B - \frac{mg}{l} A = 0, \quad B\left(\frac{mg}{l} - m\omega^2\right) = \frac{mg}{l} A$$

$$\therefore \quad \frac{A}{B} = \frac{mg - ml\omega^2}{mg} \qquad \text{...6.24}$$

From equations 6.23 and 6.24,

$$\frac{mg}{3mg - ml\omega^2} = \frac{mg - ml\omega^2}{mg}$$

$$(3mg - ml\omega^2)\,(mg - ml\omega^2) = (mg)^2$$

$$3(mg)^2 - 3m^2gl\omega^2 - m^2gl\omega^2 + (ml\omega^2) = (mg)^2$$

$$m^2l^2\omega^4 - 4m^2gl\omega^2 + 2m^2g^2 = 0$$

$$l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0, \; \omega^4 - \frac{4g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

This is a quadratic equation in ω^2.

$$\therefore \quad \omega^2 = \frac{\frac{4g}{l} \pm \sqrt{\left(\frac{4g}{l}\right)^2 - 4 \times 2\left(\frac{g}{l}\right)^2}}{2}$$

$$\therefore \quad \omega^2 = \frac{2g}{l} \pm \sqrt{\frac{16}{4}\left(\frac{g}{l}\right)^2 - \frac{4}{4} \times 2\left(\frac{g}{l}\right)^2}$$

$$\therefore \quad \omega^2 = \frac{2g}{l} \pm \frac{g}{l}\sqrt{4 - 2} \qquad \therefore \quad \omega^2 = \frac{2g}{l} \pm \sqrt{2}\,\frac{g}{l}$$

$$\therefore \quad \omega_{1n}^2 = \frac{g}{l}(2 - \sqrt{2}) \qquad \therefore \quad \omega_{2n}^2 = \frac{g}{l}(\sqrt{2} + 2)$$

$\therefore \qquad \omega_{1n}^2 = 0.59\dfrac{g}{l} \qquad\qquad \therefore\ \omega_{2n}^2 = 3.41\dfrac{g}{l}$

$\therefore \qquad \omega_{1n}^2 = 0.77\sqrt{\dfrac{g}{l}}$ rad/s $\qquad\qquad \therefore\ \omega_{2n}^2 = 1.85\sqrt{\dfrac{g}{l}}$ rad/s

To draw the mode shapes

(i) First mode shape At $\omega^2 = \omega_{1n}^2 = 0.59\dfrac{g}{l}$ in Eq. 6.23

$$\frac{A}{B} = \frac{mg}{3mg - ml\omega^2}$$

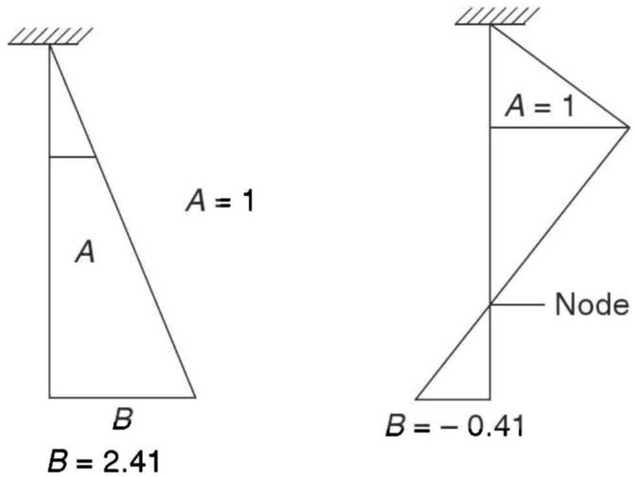
$$\therefore \qquad \frac{A}{B} = \frac{g}{3g - ml\omega^2} = \frac{g}{3g - l \times 0.59\dfrac{g}{l}} = \frac{g}{3g - 0.59\,g},$$

$$\frac{A}{B} = \frac{1}{2.41}, \quad \text{i.e. } A = 1,\ B = 2.41$$

(ii) Second mode shape At $\omega^2 = \omega_{2n}^2 = 3.41\dfrac{g}{l}$, $\dfrac{A}{B} = \dfrac{g}{3g - 3.41g}$,

$$\frac{A}{B} = \frac{1}{-0.41}, \quad \text{i.e. } A = 1,\ B = -0.41$$

The first-mode and second-mode shapes are as shown in Fig. p-6.6(d) and Fig. p-6.6(e).



(d) Same phase (e) Out phase

Fig. p-6.6 *Mode shapes*

EXAMPLE 6.7

Determine the two natural frequencies and the corresponding mode shapes of the system shown in Fig. p-6.7(a).

Solution Let at any instant give an angular displacement 'θ_1' to the mass 'm' and 'θ_2' to the mass '$2m$' from the vertical position and 'x_1' and 'x_2' from the vertical equilibrium position to the mass 'm' and '$2m$' respectively in Fig. p-6.7(a). The FBD is as shown in Fig. p-6.7(b).

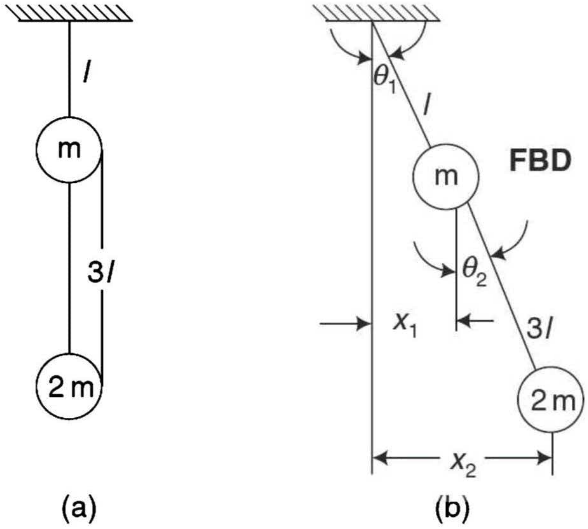


Fig. p-6.7 *Double pendulum*

Now applying Newton's second law of motion to the mass 'm', assuming that $x_2 > x_1$,

$\sin\theta_1 = \frac{x_1}{l}$, $\sin\theta_2 = \frac{x_2 - x_1}{3l}$, $\theta_1 \approx \frac{x_1}{l}$, $\theta_2 \approx \frac{x_2 - x_1}{3l}$

If 'θ_1' and θ_2 are very small, $\sin\theta_1 \approx \theta_1$ and $\sin\theta_2 \approx \theta_2$

$$T_2 \cos\theta_2 = 2mg, \quad T_1 \cos\theta_1 = mg + T_2 \cos\theta_2$$

$$T_2 \approx 2mg\ T_1 \approx 3mg, \quad m\ddot{x}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

$$m\ddot{x}_1 + \frac{11mg}{3l}x_1 - \frac{2mgx_2}{3l} = 0 \qquad ...6.25$$

$$x_2 = -T\sin\theta_2; \quad \ddot{x}_2 = \frac{9x_2}{3l} - \frac{9x_1}{3l} = 0 \qquad ...6.26$$

$$x_1 = X_1 \sin\omega t, x_2 = X_2 \sin\omega t$$

Equations 6.25 and 6.26 become $\left(\frac{119}{3l} - \omega^2\right)X_1 - \frac{29X_2}{3l} = 0$

Frequency equation $\omega^4 - \omega^2\left(\frac{48}{l}\right) + \frac{9^2}{l^2} = 0$, $-\frac{9}{3l}X_1 + \left(\frac{9}{3l} - \omega^2\right)X_2 = 0$,

$$\omega_1 = 518\sqrt{\frac{9}{l}}, \omega_2 = 1.932\sqrt{\frac{9}{l}}\ (\omega_{12})^2 = \frac{9}{l}(2 \pm \sqrt{3}) \qquad ...6.27$$

$$\left(\frac{X_1}{X_2}\right) = 0.196, -10.19 \qquad ...6.28$$

EXAMPLE 6.8

Find the natural frequencies of the system shown in Fig. p-6.8(a). Also determine the ratio of amplitudes and locate the nodes for each mode of vibration.

Draw the mode shapes also.

Solution Now at any instant give displacement 'y_1' to the mass 'm' and 'y_2' to another mass 'm' in vertical position to Fig. p-6.8(a). Then the FBD is as shown in Fig. p-6.8(b).

Now apply Newton's second law of motion to the mass 'm' assuming that $y_2 > y_1$.

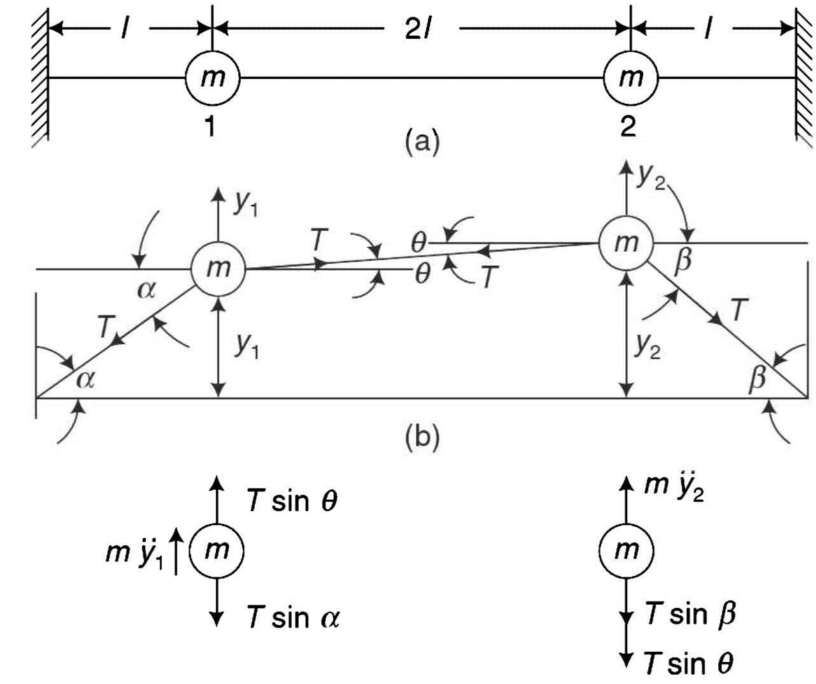


Fig. p-6.8 *System for Example 6.8*

Applying Newton's second law of motion to the mass (1),

$$\Sigma F = m\ddot{x},\ T\sin\alpha - T\sin\theta = -m\ddot{y}_1,$$

$$\therefore \qquad m\ddot{y}_1 + T\sin\alpha - T\sin\theta = 0$$

From the geometry of the FBD in Fig. p-6.8(b),

$$\sin\alpha = \frac{y_1}{l},\ \sin\beta = \frac{y_2}{l} \text{ and } \sin\theta = \frac{y_2 - y_1}{2l}$$

$$m\ddot{y}_1 + T\frac{y_1}{l} - T\left(\frac{y_2 - y_1}{2l}\right) = 0,\ m\ddot{y}_1 + T\frac{y_1}{l} - \frac{y_2}{2l} + T\frac{y_1}{2l} = 0,$$

$$m\ddot{y}_1 + \left(\frac{T}{l} + \frac{T}{2l}\right)y_1 - \frac{T_1}{2l}y_2 = 0$$

$$m\ddot{y}_1 + \frac{3}{2l}Ty_1 - \frac{T}{2l}y_2 = 0 \qquad \text{...6.29}$$

This is the differential equation of motion of the mass (1).

Applying Newton's second law of motion to the mass (2),

$$\Sigma F = m\ddot{x},\ T\sin\theta + T\sin\beta = -m\ddot{y}_2,\ m\ddot{y}_2 + T\sin\beta + T\sin\theta = 0$$

$$m\ddot{y}_2 + T\frac{y_2}{l} + T\left(\frac{y_2 - y_1}{2l}\right) = 0,\ m\ddot{y}_2 + \frac{3Ty_2}{2l} - \frac{T}{2l}y_1 = 0 \qquad \text{...6.30}$$

This is the differential equation of motion of the mass (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$y_1 = A\sin\omega t \qquad y_2 = B\sin\omega t$$

$$\ddot{y}_1 = -A\omega^2\sin\omega t \qquad \ddot{y}_2 = -B\omega^2\sin\omega t$$

Using the values of y_1, y_2 and $\ddot{y}_1$ in Eq. 6.29,

$$m(-A\omega^2\sin\omega t) + \frac{3T}{2l}A\sin\omega t - \frac{T}{2l}B\sin\omega t = 0,\ A\left[\frac{3T}{2l} - m\omega^2\right] = B\frac{T}{2l}$$

The amplitude ratio $\frac{A}{B} = \frac{T}{3T - 2ml\omega^2}$ ...6.31

Using the values of $y_1, y_2, \ddot{y}_2$ in Eq. 6.30,

$$m(-B\omega^2 \sin \omega t) + \frac{3T}{2l} B \sin \omega t - \frac{T}{2l} A \sin \omega t = 0$$

$$B\left[\frac{3T}{2l} - m\omega^2\right] = A\frac{T}{2l}$$

The amplitude ratio $\frac{A}{B} = \frac{3T - 2ml\omega^2}{T}$ ...6.32

From equations 6.31 and 6.32, $\frac{T}{3T - 2ml\omega^2} = \frac{3T - 2ml\omega^2}{T}$

$\therefore\ (3T - 2ml\omega^2)^2 = T^2,\ 3T - 2ml\omega^2 = \pm T,\ 2ml\omega^2 = 3T \pm T,\ \omega^2 = \frac{3T \pm T}{2ml}$

$\therefore\ \omega_{1n}^2 = \frac{2T}{2ml} = \frac{T}{ml} \quad \therefore\ \omega_{1n} = \sqrt{\frac{T}{ml}}$ rad/s $\therefore\ \omega_{2n}^2 = \frac{4T}{2ml} = \frac{2T}{ml}$

$\therefore\ \omega_{2n}^2 = 1.41\sqrt{\frac{T}{ml}}$ rad/s

To draw the mode shapes

At $\omega^2 = \omega_{1n}^2 = \frac{T}{ml}$ in Eq. 6.31 at $\omega^2 = \omega_{2n}^2 = \frac{2T}{ml}$ in Eq. 6.32,

$$\frac{A}{B} = \frac{T}{3T - 2ml\omega^2} \quad \frac{T}{3T - 2ml \times \frac{2T}{ml}} = \frac{T}{3T - 2ml \times \frac{T}{ml}} = 1$$

i.e. $A = 1, B = 1, \quad \frac{A}{B} = \frac{T}{3T - 4T} = -1,$ i.e. $A = 1, \quad B = -1$

The first-mode and second-mode shapes are as shown in Fig. p-6.8(c) and Fig. p-6.8(d).

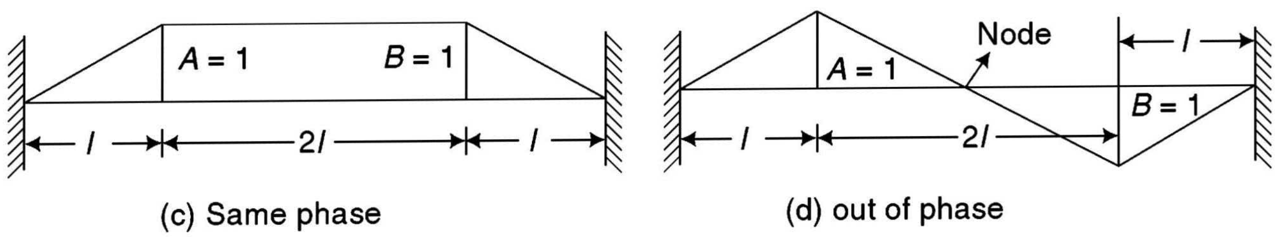


Fig. p-6.8 *Mode shape*

EXAMPLE 6.9

Determine the natural frequencies for the system shown in Fig. p-6.9(a).

Solution Let at any instant give an angular displacement 'θ_1' to the mass 'm' and 'θ_2' to the mass 'm' from the vertical position to Fig. p-6.9(a). The FBD is as shown in Fig. p-6.9(b).

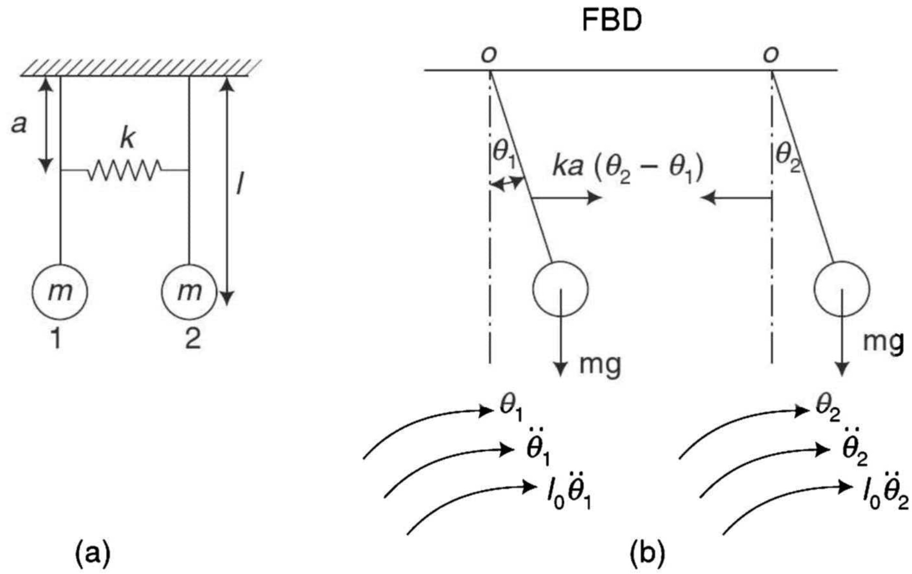


Fig. p-6.9 *Double coupled pendulum*

Now apply Newton's second law of motion to mass 'm' assuming that $\theta_2 > \theta_1$.

Applying Newton's second law of motion to the mass (1),

$$\Sigma M_o = I_0\ddot{\theta},\ ka\,(\theta_2 - \theta_1)\,a - mgl\sin\theta_1 = I_0\ddot{\theta}_1, \text{ if } \theta \text{ is very small, } \sin\theta \approx \theta$$

$$\therefore \quad I_0\ddot{\theta}_1 + mgl\,\theta_1 - ka^2\,(\theta_2 - \theta_1) = 0$$

But $\quad I_0 = ml^2,\ ml^2\ddot{\theta}_1 + mgl\,\theta_1 + ka^2\theta_1 - ka^2\theta_2 = 0$

$$ml^2\ddot{\theta}_1 + (ka^2 + mgl)\theta_1 - ka^2\theta_2 = 0 \qquad \text{...6.33}$$

This is the differential equation of motion for the mass (1).

Apply Newton's second law of motion to mass (2), $-ka\,(\theta_2 - \theta_1)\,a - mgl\sin\theta_2 = I_o\ddot{\theta}_2$, if θ is very small, $\sin\theta \approx \theta$

$$I_0\ddot{\theta}_2 + ka^2\,(\theta_2 - \theta_1) = mgl\theta_2$$

But $\quad I_0 = ml^2$

$$ml^2\ddot{\theta}_2 + ka^2\theta_2 + mgl\theta_2 - ka^2\theta_1 = 0,\ ml^2\,\ddot{\theta}_2 + (ka^2 + mgl)\,\theta_2 - ka^2\theta_1 = 0 \qquad \text{...6.34}$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A\sin\omega t \qquad \theta_2 = B\sin\omega t$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t \qquad \ddot{\theta}_2 = -B\omega^2\sin\omega t$$

Using these value of θ_1, θ_2, and $\ddot{\theta}_1$ in Eq. 6.33, we have

$$ml^2\,(-A\omega^2\sin\omega t) + (ka^2 + mgl)\,A\sin\omega t - ka^2\,B\sin\omega t = 0$$

$$A[(ka^2 + mgl) - ml^2\omega^2] = ka^2B$$

The amplitude ratio $\dfrac{A}{B} = \dfrac{ka^2}{[(ka^2 + mgl) - ml^2\omega^2]}$ ...6.35

Using these values of θ_1, θ_2, and $\ddot{\theta}_1$ in Eq. 6.34, we have

$$ml^2\,(-B\omega^2\sin\omega t) + (ka^2 + mgl)\,B\sin\omega t - ka^2\,A\sin\omega t = 0$$

$$B[(ka^2 + mgl) - ml^2\omega^2] = ka^2A$$

The amplitude ratio $\therefore \dfrac{B}{A} = \dfrac{ka^2}{[(ka^2 + mgl) - ml^2\omega^2]}$ ...6.36

From equations 6.35 and 6.36, $[(ka^2 + mgl) - ml^2\omega^2]^2 = [ka^2]^2$

$$(ka^2 + mgl) - ml^2\omega^2 = \pm\, ka^2, \qquad ml^2\omega^2 = (ka^2 + mgl) \pm ka^2$$

$$\omega^2 = \frac{ka^2 + mgl \pm ka^2}{ml^2},\ \omega_{1n}^2 = \frac{ka^2 + mgl - ka^2}{ml^2},\ \omega_{2n}^2 = \frac{ka^2 + mgl + ka^2}{ml^2},\ \omega_{1n}^2 = \frac{g}{l}$$

$$\omega_{1n} = \sqrt{\frac{g}{l}}\ \text{rad/s}, \qquad \omega_{2n}^2 = \frac{2ka^2 + mgl}{ml^2}, \qquad \omega_{2n} = \sqrt{\frac{2ka^2 + mgl}{ml^2}}\ \text{rad/s.}$$

EXAMPLE 6.10

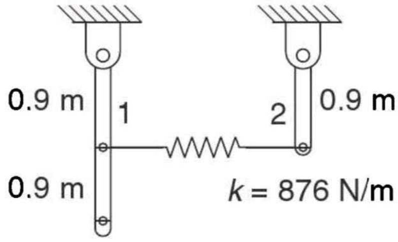


Fig. p-6.10 *Uniform rods*

Two uniform slender rods weighing w_1 = 131.4 N and w_2 = 65.7 N are suspended at their upper ends and are connected by a spring of stiffness 876 N/m as shown in Fig. p-6.10. Compute the natural frequencies of the system.

Solution Assume that at any instant during the vibratory motion the bars make angles 'θ_1' and 'θ_2' where $\theta_1 > \theta_2$. The compressive force in the spring is 876 k $(\theta_2 - \theta_1)$. The mass moment of inertia are

$$I_1 = \frac{M_1 L_1}{3} = \frac{(131.4) \times 1.8^2}{3 \times 9.81} = 14.47\ \text{kg-m}^2,\ I_2 = \frac{M_2 L_2^2}{3} = \frac{(65.7) \times 0.9^2}{3 \times 9.81} = 1.81\ \text{kg-m}^2$$

Assume that both bars swing in the same direction and using the Newton's second law of motion for the moments about the hinges,

$$I_1 \theta_1 = -w_1 \times 0.9 \times \theta_1 - k(\theta_1 - \theta_2) \times 0.9^2 \qquad \text{...6.37}$$

$$I_2 \theta_2 = -w_2 \times 0.9 \times \theta_2 - k(\theta_1 - \theta_2) \times 0.9^2 \qquad \text{...6.38}$$

Assume the solution in the form of $\theta_1 = A \sin \omega t$ and $\theta_2 = B \sin \omega t$.

We have $\ddot{\theta}_1 = -A\omega^2 \sin \omega t$ and $\ddot{\theta}_2 = -B\omega^2 \sin \omega t$.

Substituting these values into equations 6.37 and 6.38, we get

$$(0.9\, W_1 + 0.81\, k - 1\, \omega^2)\, A - (0.81\, k)\, B = 0$$

$$-(0.81\, k)\, A + (0.45\, W_2 + 0.81\, k - I_2\, \omega^2)\, B = 0$$

Substituting the values of W_1, W_2, kI_1, I_2, we have

$$(827.82 - 14.47\, \omega^2)\, A - 709.5\, B = 0 \qquad \text{...6.39}$$

$$-709.56\, A + (739.13 - 1.81\, \omega^2)\, B = 0 \qquad \text{...6.40}$$

from which we get, $\omega^4 - 465.6\, \omega^2 + 4152.48 = 0$

After solving the above quadratic equation, we get frequencies

$$\omega_1 = 3.02 \text{ and } \omega_2 = 21.37 \text{ rad/s.}$$

EXAMPLE 6.11

Figure. p-6.11 shows two equal pendulums free to rotate about the $Y–Y$ axis. A rubber hose of torsional stiffness k_t N-mm/rad couples together these pendulums. Find out the two natural frequencies and motion how the two principal modes may be started.

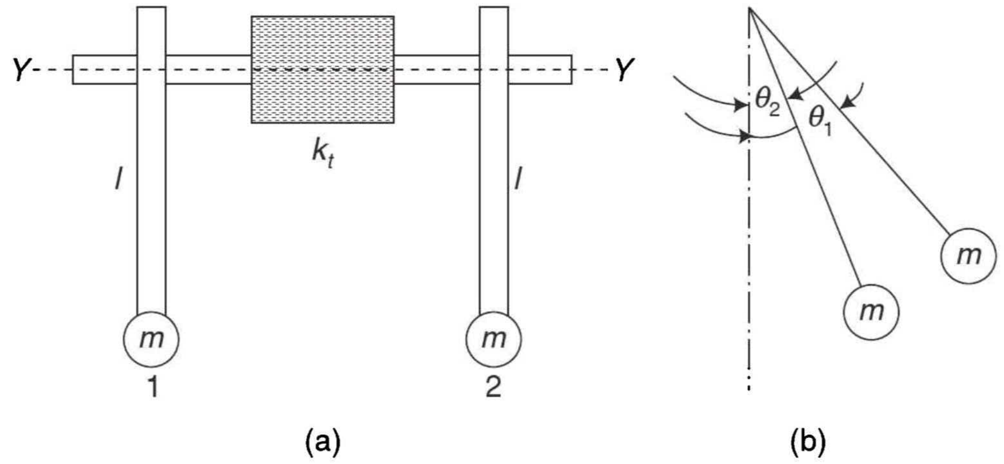


Fig. p-6.11 *Two equal pendulums*

Solution Let 'θ_1' and 'θ_2' be the angular displacements of the pendulum of mass (1) and mass (2). Applying Newton's second law of motion for mass (1) and mass (2),

$$ml^2 \ddot{\theta}_1 = -k_t (\theta_1 - \theta_2) - mgl\theta_1 \qquad ...6.41$$

is the differential equation for the mass (1).

$$ml^2 \ddot{\theta}_2 = k_t (\theta_1 - \theta_2) - mgl\theta_2 \qquad ...6.42$$

is the differential equation for the mass (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t \quad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of θ_1, θ_2, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in equations 6.41 and 6.42,

$$ml^2 \ddot{\theta}_1 + k_t (\theta_1 - \theta_2) + mgl\theta_1 = 0, \qquad ml^2 \ddot{\theta}_2 - k_t (\theta_1 - \theta_2) + mgl\theta_2 = 0$$

$$(k_t + mgl - ml^2 \omega^2) A_1 = k_t A_2, \qquad k_t A_1 = (k_t + mgl - ml^2 \omega^2) A_2$$

The amplitude ratio, $\dfrac{A_1}{A_2} = \dfrac{k_t}{k_t + mgl - ml^2 \omega^2}$, and also, $\dfrac{A_2}{A_1} = \dfrac{k_t + mgl - ml^2 \omega^2}{k_t}$

From these two above equations the frequency is obtained as

$$k_t + mgl - ml^2 \omega^2 = k_t, \quad \omega^2 = \frac{g}{l} + \frac{k_t}{ml^2}(1 \pm 1), \text{ or } \omega_{1,2} = \sqrt{\frac{g}{l} + \frac{k_t}{ml^2}(1 \pm 1)}$$

$$\therefore \qquad \omega_1 = \sqrt{\frac{g}{l}} \text{ rad/s}, \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k_t}{ml^2}} \text{ rad/s}$$

$$\frac{A_1}{A_2} = \frac{k_t}{k_t + mgl - ml^2 \dfrac{g}{l}} = 1, \text{ when } \omega_1 = \sqrt{\frac{g}{l}} \text{ rad/s}$$

with same direction and equal distance and leave to vibrate.

Also $\dfrac{A_1}{A_2} = \dfrac{k_t}{k_t + mgl - ml^2\left(\dfrac{g}{l} + \dfrac{2k_t}{ml^2}\right)} = -1$, when $\omega_2 = \sqrt{\dfrac{g}{l} + \dfrac{2k_t}{ml^2}}$, giving equal and opposite angular displacements to the bobs.

EXAMPLE 6.12

Determine the natural frequency for the system shown in Fig. p-6.12(a) and draw the mode shapes and locate the node for each mode of vibration. Given $I_1 = I, I_2 = 2I, \quad kt_1 = kt_2, kt_2 = kt$.

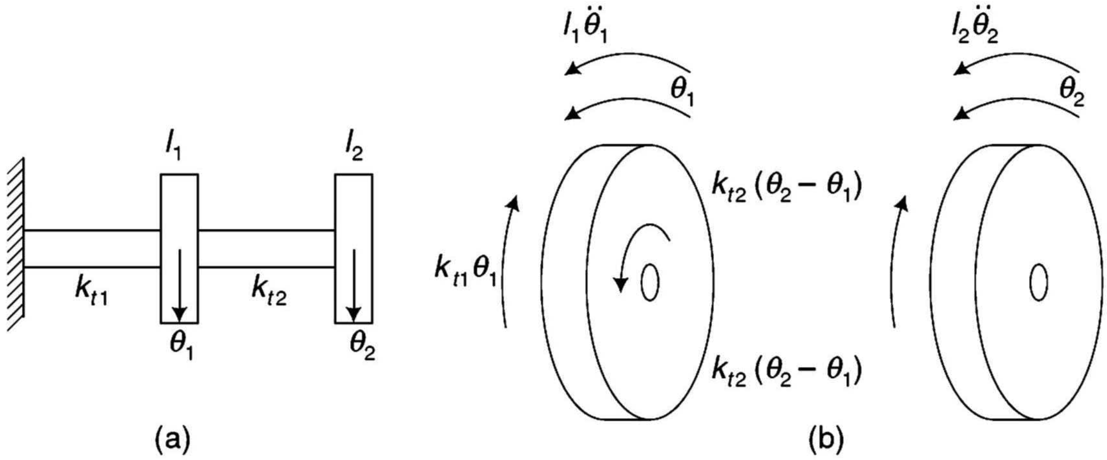


Fig. p-6.12 *System for Example 6.12*

Solution Let us compare at any instant 'θ_1' and 'θ_2' be the angular displacement of flywheels 'I_1' and 'I_2' respectively. Let 'k_t' is the torsional stiffness of the connecting shaft of flywheel 'I_1' and 'I_2', then the FBD as shown in Fig. p-6.12(b).

Applying Newton's second law of motion to disc (I_1), let $\theta_2 > \theta_1$

$$\Sigma M = I\ddot{\theta},\ kt_2(\theta_2 - \theta_1) - kt_1\theta_1 = I_1\ddot{\theta}_1,\ I_1\ddot{\theta}_1 + kt_1\theta_1 - kt_2(\theta_2 - \theta_1) = 0$$

$$I\ddot{\theta}_1 + 2kt\theta_1 - kt\theta_2 + kt\theta_1 = 0,\ I\ddot{\theta}_1 + 3kt\theta_1 - kt\theta_2 = 0 \qquad \text{...6.43}$$

This is a differential equation of motion to the disc (I_1).

Applying Newton's second law of motion to the disc (I_2),

$$\Sigma M = I\ddot{\theta},\quad -kt_2(\theta_2 - \theta_1) = I_2\ddot{\theta}_2 \quad \therefore\ I_2\ddot{\theta}_2 + kt_2\theta_2 - kt_2\theta_1 = 0$$

$$2I\ddot{\theta}_2 + kt\theta_2 - kt\theta_1 = 0 \qquad \text{...6.44}$$

This is a differential equation of motion to the disc (I_2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies.

Let one of these components be,

$$\theta_1 = A_1 \sin \omega t, \qquad \theta_2 = A_2 \sin \omega t$$

$$\ddot{\theta}_1 = -A_1\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -A_2\omega^2 \sin \omega t$$

Substituting these values in equations 6.43 and 6.44, we get

$$\therefore \qquad I(-A\omega^2) \sin \omega t + 3kt\, A \sin \omega t - kt\, B \sin \omega t = 0$$

Amplitude ratio $\dfrac{A}{B} = \dfrac{kt}{3kt - I\omega^2}$ ...6.45

$$2I(-B\omega^2 \sin \omega t) + kt\, B \sin \omega t - A\, kt \sin \omega t = 0$$

$\therefore \qquad \dfrac{A}{B} = \dfrac{kt - 2I\omega^2}{kt}$ ...6.46

From equations 6.45 and 6.46, $\dfrac{kt}{3kt - I\omega^2} = \dfrac{kt - 2I\omega^2}{kt}$.

$$2I^2\omega^4 - 7kt\,\omega^2 I + 2kt^2 = 0, \; \omega^4 - \frac{7kt}{2I} + \frac{kt^2}{I^2} = 0.$$

This is a quadratic equation in ω^2

$$\therefore \qquad \omega^2 = \frac{\dfrac{7kt}{2I} \pm \sqrt{\left(\dfrac{-7kt}{2I}\right)^2 - \dfrac{4kt^2}{I^2}}}{2}$$

$$\therefore \qquad \omega_{1n}^2 = 0.315\,\frac{kt}{I}, \quad \omega_{2n}^2 = 3.185\,\frac{kt}{I}$$

$$\omega_{1n} = 0.56\sqrt{\frac{kt}{I}} \text{ rad/s}, \quad \omega_{2n} = 1.78\sqrt{\frac{kt}{I}} \text{ rad/s}$$

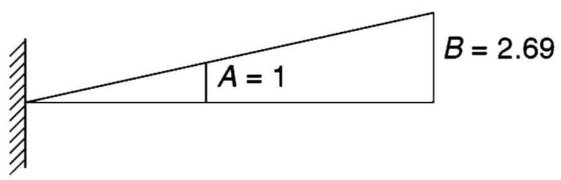
To draw the mode shapes

At $\omega^2 = \omega_{1n}^2 = 0.315\,\dfrac{kt}{I}$ in Eq. 6.45,

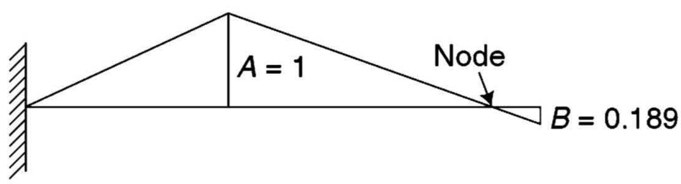
$$\frac{A}{B} = \frac{kt}{3kt - I \times 0.315 \times \dfrac{kt}{I}} = \frac{1}{2.69}$$

$$\therefore \qquad A = 1, B = 2.69$$

at $\omega^2 = \omega_{2n}^2 = 3.185\,\dfrac{kt}{I}$, $\dfrac{A}{B} = \dfrac{kt}{3kt - I \times 3.185\,\dfrac{kt}{I}} = \dfrac{1}{-0.189}$ $\therefore A = 1, B = -0.189$



(c) First mode shape



(d) Second mode shape

Fig. p-6.12 *Mode shapes*

EXAMPLE 6.13

Find the natural frequencies of the system shown in Fig. p-6.13(a). Also determine the ratio of amplitudes and the mode shapes. Given $I_1 = I_0$, $I_2 = 2I_0$, and $k_{t1} = k_{t2} = k_{t3} = kt$.

Solution Let 'θ_1' and 'θ_2' be the angular displacement of the disc (I_1) and the disc (I_2) respectively. Then the FBD is as shown in Fig. p-6.13(b). Then the equation of motion for the disc (I_1) may be written as

$$I_1\ddot{\theta}_1 = -kt_1\theta_1 + kt_2(\theta_2 - \theta_1), \quad I_1\ddot{\theta}_1 + kt\theta_1 + kt(\theta_1 - \theta_2) = 0$$

$$I_1\ddot{\theta}_1 + 2k_1\theta_1 - kt\theta_2 = 0 \qquad \text{...6.47}$$

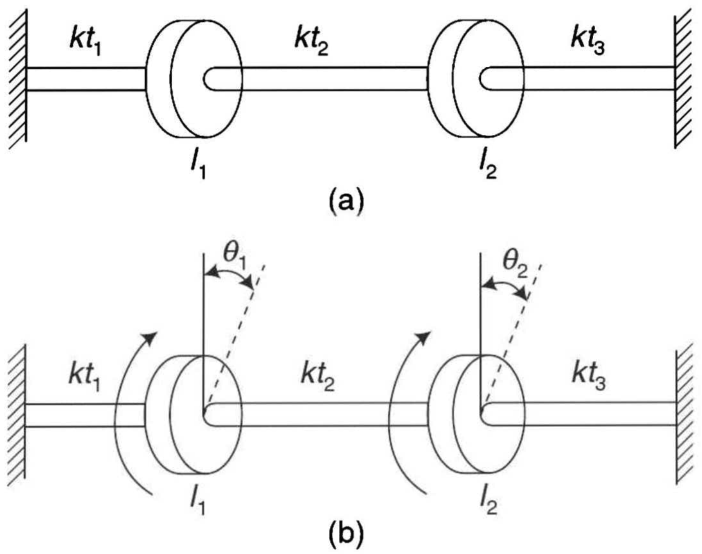


Fig. p-6.13 *System for Example 6.13*

Similarly, the equation of motion for the disc (I_2) may be written as

$$I_2\ddot{\theta}_2 = -kt_2(\theta_2 - \theta_1) - kt_3\theta_2, \quad I_2\ddot{\theta}_2 + kt(\theta_2 - \theta_1) + kt\theta_2 = 0$$

$$I_2\ddot{\theta}_2 + kt\theta_2 + kt\theta_2 - kt\theta_1 = 0, \quad I_2\ddot{\theta}_2 + 2kt\theta_2 - kt\theta_1 = 0$$

$$2I_2\ddot{\theta}_2 + 2kt\theta_2 - kt\theta_1 = 0 \qquad \text{...6.48}$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A_1 \sin \omega t, \qquad \theta_2 = A_2 \sin \omega t$$

$$\ddot{\theta}_2 = -A_1\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -A_2\omega^2 \sin \omega t$$

Using these values in equations 6.47 and 6.48, we have

$$-\omega^2 I_0 A_1 + 2ktA_1 - ktA_2 = 0, \; (-\omega^2 I_0 + 2kt)A_1 - ktA_2 = 0$$

$$(-2\omega^2 I_0 + 2kt)A_2 - ktA_1 = 0$$

The amplitude ratios are given by

$$\frac{A_1}{A_2} = \frac{kt}{-\omega^2 I_0 + 2kt} = \frac{(-2\omega^2 I_0 + 2kt)}{kt} \qquad \text{...6.49}$$

The frequency equation can be written as $2(-\omega^2 I_0 + 2kt)(-\omega^2 I_0 + kt) - kt^2 = 0$,

$$2(\omega^4 I_0^2 - \omega^2 I_0 kt + 2kt\omega^2 I_0 + 2kt^2) - kt^2 = 0,\ 2\omega^4 I_0^2 - 6\ ktI_0\omega^2 + 3kt^2 = 0$$

$$\omega^4 - \frac{3kt}{I_0}\omega^2 + \frac{3kt^2}{2I_0^2} = 0,\ \omega^2 = \frac{\frac{3kt}{I_0} \pm \sqrt{\left(\frac{3kt}{I_0}\right)^2 - 4 \cdot \frac{3}{2}\frac{kt^2}{I_0^2}}}{2} = \frac{1.5kt}{I_0} + \frac{\sqrt{3}}{2}\frac{kt^2}{I_0}$$

So $$\omega_1 = 1.5\sqrt{\frac{kt}{I_0}}\ \omega_2 = 0.80\sqrt{\frac{kt}{I_0}}$$

The amplitude ratios are given by, $\left(\frac{A_1}{A_2}\right)_{\omega 1} = \frac{kt}{-\omega_1^2 I_0 + 2kt} = \frac{kt}{-\left(1.5\sqrt{\frac{kt}{I_0}}\right) I_0 + 2kt} = -4$

$$\left(\frac{A_1}{A_2}\right)_{\omega 2} = \frac{-\omega_1^2 I_0 + 2kt}{kt} = \frac{-2(0.8)^2 \frac{kt}{I_0} I_0 + 2kt}{kt} = \frac{-2 \times 0.64 + 2}{1} = +0.72$$

Mode shapes are as shown in Fig. p-6.13(b) and Fig. p-6.13(c).

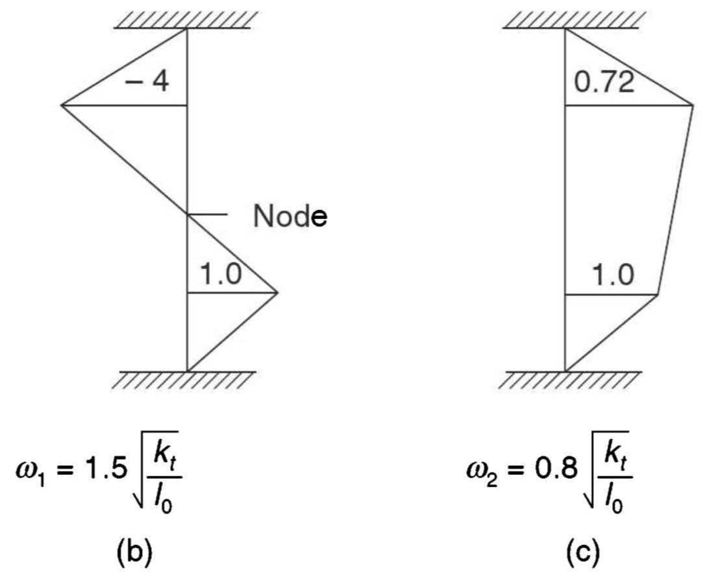


Fig. p-6.13 *Mode shapes*

EXAMPLE 6.14

Derive the frequency equation for the pulley-mass system shown in Fig. p-6.14(a). The pulley has a mass of 'M' and effective radius of 'R'. Assume that the cord, which passes over the pulley, does not slip, if $k_1 = 60$ N/m, $k_2 = 40$ N/m, $m = 2$ kg and $M = 10$ kg. Determine the natural frequencies and mode shapes.

Solution Let us at any instant give a vertical displacement 'x' to the mass 'm' as shown in Fig. p-6.14(a). Since there is no slip between the card and cylinder of mass 'M', so the vertical displacement 'x' causes the cylinder to rotate by an angle 'θ' as shown in FBD of Fig. p-6.14(b).

Now applying Newton's second law of motion to 'm' (rectilinear motion),

$$\Sigma F = ma, \quad m\ddot{x} = -k(x - R\theta)$$

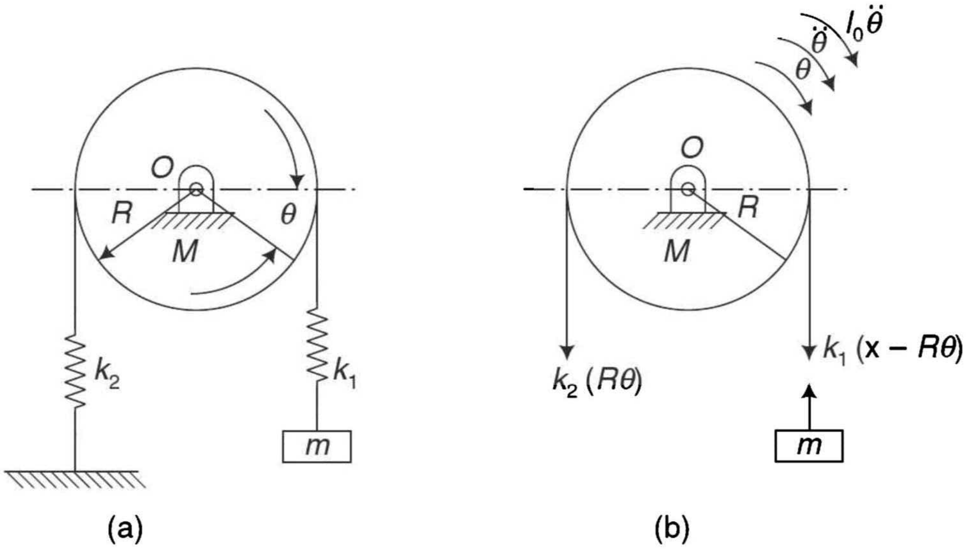


Fig. p-6.14 *Pulley-mass system*

For mass M (rotational), $J\ddot{\theta} = \Sigma T$

But $$J = \frac{MR^2}{2}$$

$$\frac{MR^2}{2}\ddot{\theta} = -k_1\,(R\theta - x)\,R - k_2\,(R\theta)\,R \qquad \text{...6.50}$$

$$J\ddot{\theta} + k_1R^2\theta - k_1Rx + k_2R^2\theta = 0 \qquad \text{...6.51}$$

This is the differential equation of motion of mass and pulley

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x = \sin \omega t, \qquad \theta = B \sin \omega t$$

$$\ddot{x} = -A\omega^2 \sin \omega t, \quad \ddot{\theta} = -B\omega^2 \sin \omega t$$

Using these values in Eq. 6.50, we have

$$-JBw^2 \sin \omega t + k_1\,R^2B \sin \omega t - k_1RA \sin \omega t + k_2R^2\,B \sin \omega t = 0\,,\ \sin \omega t \neq 0$$

$$-JB\omega^2 + k_1R^2B - k_1RA + k_2\,R^2B = 0$$

$$-k_1RA + [k_1R^2 + k_2R^2 - J\omega^2]\,B = 0, \quad k_1RA = \left[k_1R^2 + k_2R^2 - \frac{MR^2}{2}\cdot\omega^2\right]B$$

$$\frac{A}{B} = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega^2\right]R^2}{k_1R} = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega^2\right]R}{k_1}$$

The frequency equation is equating to two equations ratio,

$$\left(\frac{Mm}{2}\right)R^2\omega^4 - \left[\left(\frac{MR^2}{2}\right)k_1 + m\,(k_1 + k_2)\,R^2\right]\omega^2 + k_1k_2R^2 = 0$$

Dividing by $MmR^2/2$

$$\omega^4 - \left[\frac{k_1}{m} + \frac{2(k_1 + k_2)}{M}\right]\omega^2 + \frac{2k_1k_2}{Mm} = 0 \qquad \text{...6.52}$$

This is in the form of a quadratic equation.

The solution is given by the roots of the equation as

$$\omega_{1,2}^2 = \frac{\left[\frac{k_1}{m} + \frac{2(k_1+k_2)}{M}\right] \pm \sqrt{\left[\frac{k_1}{m} + \frac{2(k_1+k_2)}{M}\right]^2 - \frac{4 \times 2k_1k_2}{Mm}}}{2 \times 1}$$

$$\omega_{1,2}^2 = \frac{\left[\frac{60}{2} + \frac{2(60+40)}{10}\right] \pm \sqrt{\left[\frac{60}{2} + \frac{2(60+40)}{10}\right]^2 - \frac{4 \times 2 \times 60 \times 40}{10 \times 2}}}{2}$$

$$\omega_{1,2}^2 = \frac{50 \pm 39.24}{2}$$

$$\omega_1^2 = \frac{50 + 39.24}{2} = 44.62$$

$$\therefore \quad \omega_1 = \sqrt{44.62} = 6.68 \text{ rad/s}$$

$$\omega_2^2 = \frac{50 - 39.24}{2} = 5.38, \ \omega_2 = \sqrt{5.38} = 2.319 \text{ rad/s}$$

The amplitude ratios are

$$\left(\frac{A}{B}\right)_1 = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega_1^2\right]R}{k_1} = \frac{\left[60 + 40 - \left(\frac{10}{2}\right) \times 44.62\right]1}{k60} = -2.052$$

$$\left(\frac{A}{B}\right)_2 = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega_2^2\right]R}{k_1} = \frac{\left[60 + 40 - \left(\frac{10}{2}\right)5.38\right] \times 1}{60} = 1.22$$

The first-mode and second-mode shapes are as shown in Fig. p-6.14(c) and Fig. p-6.14(d).

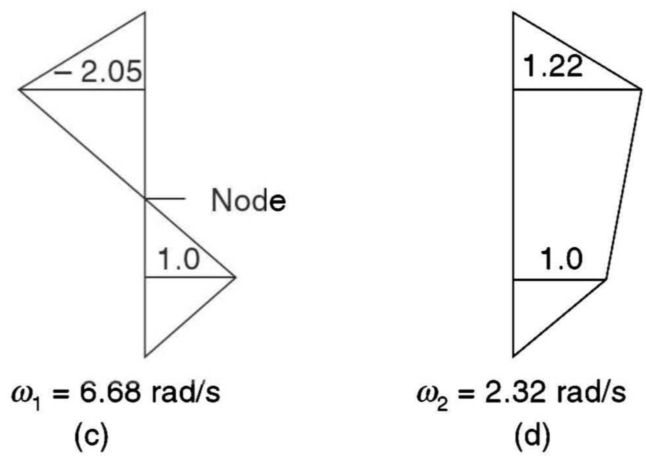


Fig. p-6.14 *Mode shapes*

EXAMPLE 6.15

Determine the frequency equation for the system shown in Fig. p-6.15(a) and determine the natural frequencies if $k_1 = k_2 = k_3 = k$, $m_1 = m_2 = m$ and $r_1 = r_2 = r$

Solution Let us at any instant give an angular displacement 'θ_1' to the mass 'm_1' and 'θ_2' of the mass 'm_2' as shown in Fig. p-6.15(a). Then the FBD is as shown in Fig. p-6.15(b). Let $\theta_2 > \theta_1$.

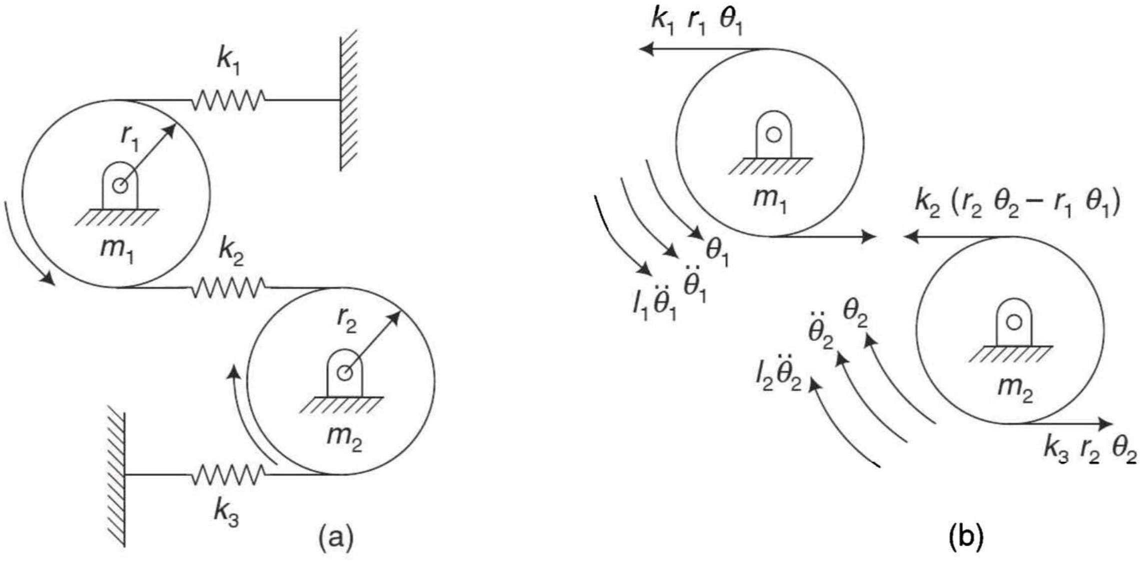


Fig. p-6.15 *Pulley system*

Applying Newton's second law of motion to the disc (1),

$$\Sigma M = I\ddot{\theta},\ k_2(r_2\theta_2 - r_1\theta_1)\, r_1 - k_1 r_1 \theta_1 \cdot r_1 = I_1 \ddot{\theta}_1$$

$\therefore$ $$I_1\ddot{\theta}_1 + k_1 r_1^2\, \theta_1 - k_2 r_1\,(r_2\theta_2 - r_1\theta_1) = 0$$

$\therefore$ $$\frac{mr^2}{2}\,\ddot{\theta}_1 + k_1 r^2\theta_1 - k_2 r^2\theta_2 + k_2 r^2\theta_1 = 0$$

$\therefore$ $$\frac{m}{2}\,\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$

$\therefore$ $$m\ddot{\theta}_1 + 4k\theta_1 - 2k\theta_2 = 0 \qquad \text{...6.53}$$

This is the differential equation of motion for the disc (1).

Applying Newton's second law of motion to the disc (2), $\Sigma M = I\ddot{\theta}$,

$$k_2(r_2\theta_2 - r_1\theta_1)\, r_2 + k_3 r_2\theta_2 r_2 = -I_2\ddot{\theta}_2$$

$\therefore$ $$I_2\ddot{\theta}_2 + k_2 r_2\,(r_2\theta_2 - r_1\theta_1) + k_3 r_2^2\theta_2 = 0$$

$\therefore$ $$\frac{mr^2}{2}\,\ddot{\theta}_2 + kr^2\theta_2 - kr^2\theta_1 + kr^2\theta_2 = 0$$

$$\frac{m}{2}\,\ddot{\theta}_2 + 2k\theta_2 - k\theta_1 = 0$$

$\therefore$ $$m\ddot{\theta}_2 + 4k\theta_2 - 2k\theta_1 = 0 \qquad \text{...6.54}$$

This is the differential equation of motion of the disc (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of 'θ_1', 'θ_2' and $\ddot{\theta}_1$ in Eq. 6.52,

$$-m\omega^2 A + 4kA - 2kB = 0, \quad A\,(4k - m\omega^2) = 2kB$$

$\therefore$ $$\frac{A}{B} = \frac{2k}{4k - m\omega^2} \qquad \text{...6.55}$$

Using the values of 'θ_1', 'θ_2' and $\ddot{\theta}_2$ in Eq. 6.54,

$$-m\omega^2 B + 4kB - 2kA = 0, \quad B\,(4k - m\omega^2) = 2kA$$

$$\therefore \quad \frac{A}{B} = \frac{4k - m\omega^2}{2k} \qquad \text{...6.56}$$

From equations 6.55 and 6.56,

$$\frac{2k}{4k - m\omega^2} = \frac{4k - m\omega^2}{2k}$$

$$\therefore \quad (4k - m\omega^2)^2 = (2k)^2, \quad 16k^2 - 8mk\omega^2 + m^2\omega^4 = 4k^2$$

$$\therefore \quad m^2\omega^4 - 8mk\omega^2 + 12k^2 = 0$$

$$\omega^4 - 8\frac{k}{m}\omega^2 + 12\frac{k^2}{m^2} = 0$$

This is the frequency equation and this is the quadratic equation in ω^2.

$$\therefore \quad \omega^2 = \frac{\dfrac{8k}{m} \pm \sqrt{\left(\dfrac{8k}{m}\right)^2 - 4\dfrac{12k^2}{m^2}}}{2}$$

$$\therefore \quad \omega^2 = \frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}$$

$$\therefore \quad \omega^2 = \frac{4k}{m} \pm \frac{2k}{m}$$

$$\therefore \quad \omega_{1n}^2 = \frac{2k}{m}, \; \omega_{2n}^2 = \frac{6k}{m}$$

$$\omega_{1n} = 1.41\sqrt{\frac{k}{m}} \text{ rad/s}, \; \omega_{2n} = 2.45\sqrt{\frac{k}{m}} \text{ rad/s}$$

where ω_{1n} and ω_{2n} are first and second natural frequencies.

EXAMPLE 6.16

Determine the frequency equation for the system as shown in Fig. p-6.16(a).

Solution Let us at any instant give a displacement 'θ' to the mass 'M' and the attached mass 'm' as shown in Fig. p-6.16(a). Then the FBD as shown in Fig. p-6.16(b).

From the geometry of Fig. p-6.16(b), let $\theta > \phi$

$$X = r\phi, \quad y = x + l\sin\theta, \quad y = x + l\theta \text{ and } \ddot{y} = \ddot{x} + l\ddot{\theta}$$

For small angles of 'θ', let us assume the cylinder is oscillating about the point '0'. $\cos\theta \approx 1$, $\sin\theta \approx \theta$.

Considering mass 'm', $\Sigma V = 0$

$$T\cos\theta - mg = 0, \; T = mg, \text{ if '}\theta\text{' is very small } \cos\theta = 1$$

Applying Newton's second law of motion to cylinder, $\Sigma M_0 = I_0 \ddot{\phi}$

$\therefore$ $$2kx.r - T\sin\theta.r = I_0\ddot{\phi}, \qquad I_0\ddot{\phi} + 2kr\phi.r - mg\,\theta.r = 0$$

where $$I_0 = I_G + Mr^2 = 1/2\,Mr^2 + Mr^2, \quad I_0 = 3/2\,Mr^2$$

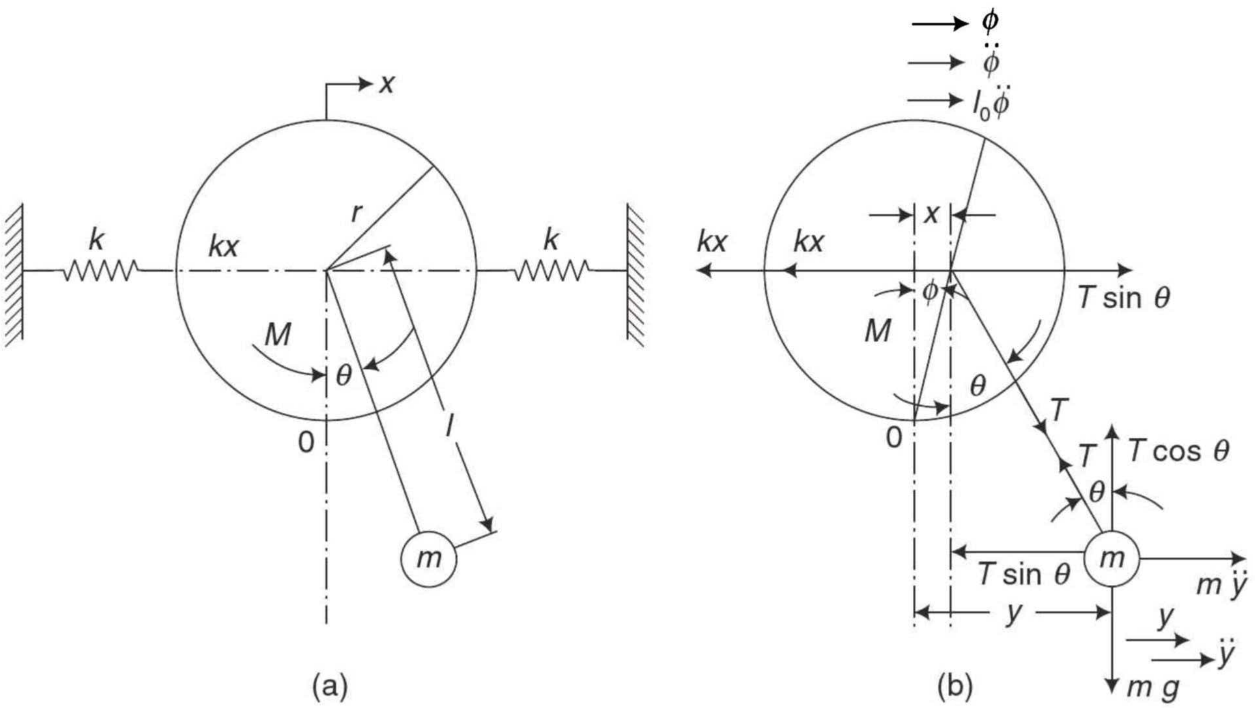


Fig. p-6.16 *System for Example 6.16*

$$\frac{3}{2}Mr^2\ddot{\phi} + 2kr^2\phi - mgr\theta = 0$$

$$\frac{3}{2}Mr\,\ddot{\phi} + 2kr\phi - mg\theta = 0 \qquad ...6.57$$

This is the differential equation of motion for the cylinder.

Applying Newton's second law of motion to the mass 'm',

$$\Sigma F = m\ddot{x}, -T\sin\theta = m\ddot{y}$$

$\therefore$ $$m\ddot{y} + T\theta = 0$$

$$m\ddot{x} + ml\ddot{\theta} + mg\theta = 0,\ l\ddot{\theta} + g\theta + \ddot{x} = 0,\ l\ddot{\theta} + g\theta + r\ddot{\phi} = 0 \qquad ...6.58$$

This is the second differential equation of motion for the bob.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\phi = A\sin\omega t, \qquad \theta = B\sin\omega t$$

$$\ddot{\phi} = -A\omega^2\sin\omega t, \qquad \ddot{\theta} = -B\omega^2\sin\omega t$$

Using the values of, $\ddot{\phi}$ and $\ddot{\theta}$ in Eq. 6.57,

$$\frac{3}{2}Mr(-A\omega^2) + 2krA - mgB = 0, \quad A\left[2kr - \frac{3}{2}Mr\omega^2\right] = mgB$$

$$\frac{A}{B} = \frac{2mg}{r[4k - 3M\omega^2]} \qquad ...6.59$$

Using the values of $\ddot{\theta}$, θ and $\ddot{\phi}$ in Eq. 6.58,

$$l(-B\omega^2) + gB - A\,\omega^2 r = 0, \quad B(g - \omega^2 l) = A\,\omega^2 r, \quad \frac{A}{B} = \frac{g - \omega^2 l}{r\omega^2} \qquad \text{...6.60}$$

From equations 6.59 and 6.60, $\dfrac{2mg}{r[4k - 3M\omega^2]} = \dfrac{g - \omega^2 l}{r\omega^2}$

$$(g - \omega^2 l)\,(4k - 3M\omega^2) = 2m\,\omega^2 g,\ 4kg - 3Mg\,\omega^2 - 4k\,l\,\omega^2 - 2mg\,\omega^2 + 3M\,l\,\omega^4 = 0$$

$$3Ml\omega^4 - [3Mg + 4kl + 2mg]\,\omega^2 + 4kg = 0,$$

This is the frequency equation for the given system.

EXAMPLE 6.17

Two identical cylinders are linked together as shown in Fig. p-6.17(a). Determine the natural frequencies of the system.

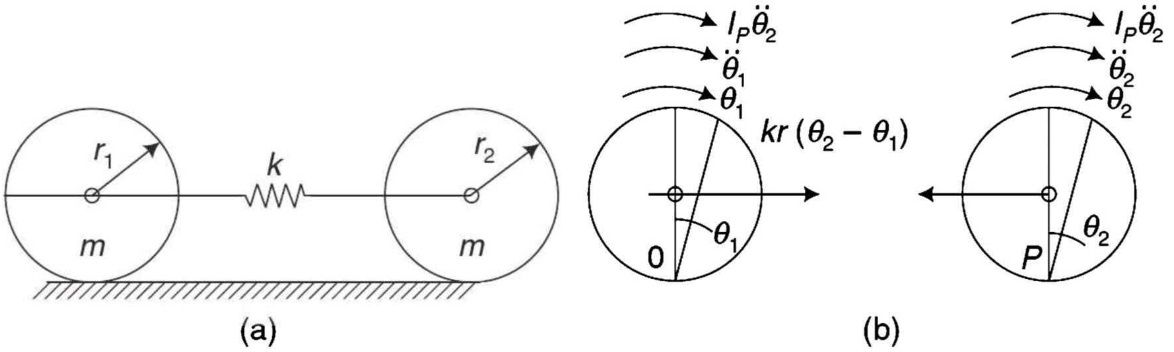


Fig. p-6.17 *Cylinder system*

Solution Let us at any instant give an angular displacement 'θ_1' to the first cylinder of mass 'm' and 'θ_2' to the second cylinder of mass 'm' as shown in Fig. p-6.17(a). Then FBD is as shown in Fig. p-6.17(b).

Applying Newton's second law of motion to the cylinder (1), let $\theta_2 > \theta_1$

$$\Sigma M_P = I_P\ddot{\theta}, -kr(\theta_2 - \theta_1)\, r = -I_P\ddot{\theta}, \quad -I_P\ddot{\theta} + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$-\frac{3}{2}mr^2\ddot{\theta}_1 + kr^2\theta_2 - kr^2\theta_1 = 0, \quad -\frac{3}{2}m\ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0 \quad ...6.65$$

This is the differential equation of motion for the cylinder (1).

Applying Newton's second law of motion to the cylinder (2),

$$\Sigma M_0 = I_0\ddot{\theta}, \quad kr(\theta_2 - \theta_1)\, r = -I_0\ddot{\theta}_2, \quad I_0\ddot{\theta}_2 + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$\frac{3}{2}m\ddot{\theta}_2 + k\theta_2 - k\theta_1 = 0 \quad ...6.66$$

This is the differential equation of motion for the cylinder (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of θ_1, θ_2, $\ddot{\theta}_1$ in Eq. 6.65,

$$\frac{3}{2} m(-A\omega^2) + kA - kB = 0, \quad A\left(k - \frac{3}{2} m\omega^2\right) = kB, \quad \frac{A}{B} = \frac{2k}{2k - 3m\omega^2} \qquad ...6.67$$

Using the values of θ_1, θ_2, $\ddot{\theta}_2$ in Eq. 6.66,

$$\frac{3}{2} m(-B\omega^2) + kB - kA = 0, \quad B\left(k - \frac{3}{2} m\omega^2\right) = kA, \quad \frac{A}{B} = \frac{2k - 3m\omega^2}{2k} \qquad ...6.68$$

From equations 6.67 and 6.68,

$$\frac{2k}{2k - 3m\omega^2} = \frac{2k - 3m\omega^2}{2k}, \; (2k - 3m\omega^2)^2 = (2k)^2, \; 2k - 3m\omega^2 = \pm 2k, \; 3m\omega^2 = 2k \pm 2k$$

$$\omega_{1n}^2 = 0, \quad \omega_{1n} = 0 \;\; \omega_{2n}^2 = \frac{4k}{3m}, \quad \omega_{2n} = \sqrt{\frac{4k}{3m}} \text{ rad/s}$$

Since one of the natural frequencies is zero, the system is a semidefinite system.

EXAMPLE 6.18

Two flywheels of moment of inertia 'I_1' and 'I_2' are keyed to the ends of a steel shaft. Derive an expression for the frequency of free torsional vibrations of the system shown in Fig. p-6.18(a) and describe the modes.

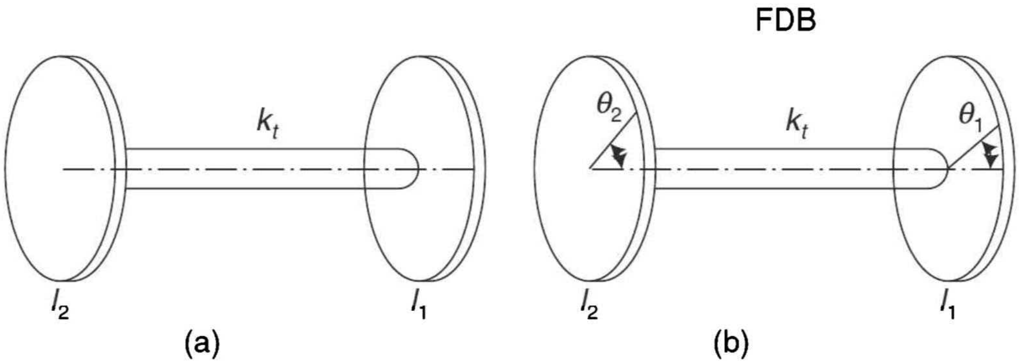


Fig. p-6.18 *Flywheel system*

Solution Let at any instant 'θ_1' and 'θ_2' be the angular displacement of flywheels 'I_1' and 'I_2' respectively. Let 'k_t' is the torsional stiffness of the connecting shaft of flywheel 'I_1' and 'I_2'. Then the FBD is as shown in Fig. p-6.18(b); assume $\theta_1 > \theta_2$

Then twist of the shaft = $\theta_1 - \theta_2$.

Then equation of motion is,

$$I_1\ddot{\theta}_1 = -k_t(\theta_1 - \theta_2) \qquad ...6.69$$

$$I_2\ddot{\theta}_2 = k_t(\theta_1 - \theta_2) \qquad ...6.70$$

Assuming that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Substituting these values of θ_1, θ_2, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in equations 6.69 and 6.70,

$$(k_t - \omega^2 I_1)\, A_1 = k_t A_2,\ (k_t - \omega^2 I_2)\, A_2 = k_t A_1$$

From these two equations, we obtained the amplitude ratios as

$$\frac{A_1}{A_2} = \frac{k_t}{k_t - \omega^2 I_1} = \frac{k_t - \omega^2 I_2}{k_t} \qquad ...6.71$$

which gives the frequency equation as

$$(k_t - \omega^2 I_1)\,(k_t - \omega^2 I_2) = k_t^2, \quad \omega^2\,[I_1 I_2 \omega^2 - k_t\,(I_1 + I_2)] = 0$$

$$\omega_1 = 0 \text{ rad/s and } \omega_2 = \sqrt{\frac{k_t\,(I_1 + I_2)}{I_1 I_2}} \text{ rad/s.}$$

Since one of the natural frequencies is zero, the system is a semidefinite system.

Substituting these values of the natural frequencies in the amplitude ratio A_1/A_2 in the above equation 6.71, we obtained the two conditions for the principal modes as shown in Fig. p-6.18(c).

$$\frac{A_1}{A_2} = \frac{k_t - \omega^2 I_2}{k_t} = 1 \text{ when } \omega_1 = 0 \text{ and } \frac{A_1}{A_2} = \frac{-I_2}{I_1}, \text{ when } \omega_2 = \sqrt{\frac{k_t\,(I_1 + I_2)}{I_1 I_2}} \text{ rad/s.}$$

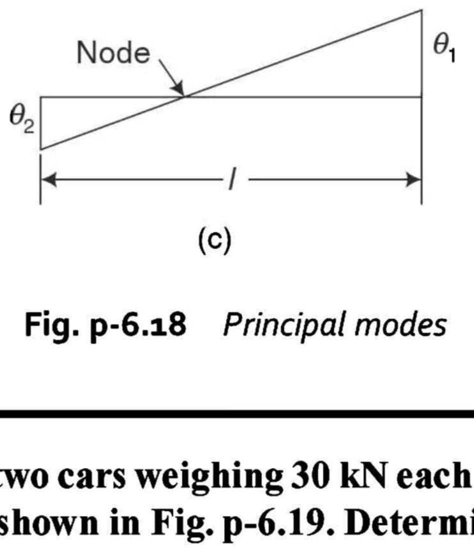


Fig. p-6.18 *Principal modes*

EXAMPLE 6.19

An electric train made of two cars weighing 30 kN each has got a spring coupling of 3000N/mm stiffness as shown in Fig. p-6.19. Determine the natural frequency of vibration of the system.

Solution For small displacements x_1 and x_2 of the two cars, applying Newton's second law of motion, we get

$$m\ddot{x}_1 = k\,(x_2 - x_1) \qquad ...6.72$$

$$m\ddot{x}_2 = -k\,(x_2 - x_1) \qquad ...6.73$$

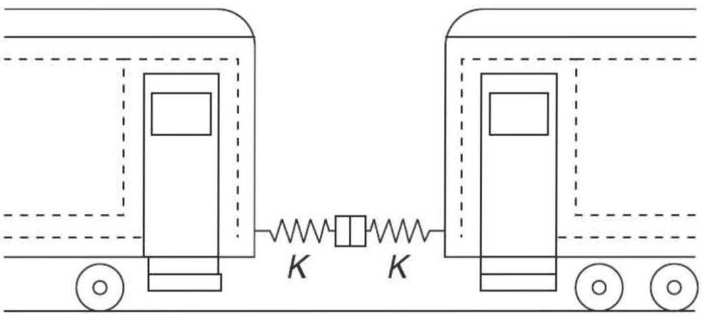


Fig. p-6.19 *Electric train*

This is the differential equation for the motion

where k = Stiffness of coupling spring, m = Mass of each car

For principal mode of vibration, let $x_1 = X_1 \sin \omega t$, $x_2 = X_2 \sin \omega t$

Putting these values of x's and their derivatives above equations 6.72 and 6.73, we get,

$$X_1(k - m\omega^2) = kX_2, \quad kX_1 = X_2(k - m\omega^2)$$

The amplitude ratios are $\dfrac{X_1}{X_2} = \dfrac{k}{k - m\omega^2} = \dfrac{k - m\omega^2}{k}$

So, the frequency equation is,

$$k^2 = (k - m\omega^2)^2, \text{ or } \omega^2 = \frac{2k}{m}, \ \omega_n^2 = \frac{2 \times 3000}{30000} = 0.2$$

$$\therefore \quad \omega_n = 0.447 \text{ rad/s}$$

$$\therefore \quad f_n = \frac{\omega_n}{2\pi} = \frac{0.447}{2\pi} = 0.07 \text{ Hz}$$

Suppose at any instant, the body be displaced through a linear distance 'x' and angular distance 'θ' as shown in Fig. 6.3(b).

Now assuming that at this instant, taking 'θ' to be small, the springs 'k_1' and 'k_2' will be compressed through an amount of $(x - l_1\theta)$ and $(x + l_2\theta)$ respectively from the equilibrium as shown. Then the FBD of entire system is as shown in Fig. 6.3(c).

Now apply Newton's second law of motion and write down the differential equations of motion for the system by considering the 'x' and 'θ' direction by taking the forces and the moments in the respective directions acting on the system.

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta), \quad J\ddot{\theta} = k_1 l_1(x - l_1\theta) - k_2 l_2(x + l_2\theta) \text{ or}$$

$$\left.\begin{aligned} m\ddot{x} + x\,(k_1 + k_2) &= (k_1 l_1 - k_2 l_2)\theta \\ J\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\,\theta &= (k_1 l_1 - k_2 l_2)\,x \end{aligned}\right\} \quad ...6.74$$

$$\text{Put} \quad \begin{bmatrix} P = \dfrac{k_1 + k_2}{m} \\ Q = \dfrac{k_1 l_1 - k_2 l_2}{m} \\ R = \dfrac{k_1 l_1^2 + k_2 l_2^2}{J} \end{bmatrix} \quad ...6.75$$

Also we known that $J = mr^2$

Substitute all these values in equations 6.74 and it will

$$\text{reduce to} \quad \begin{bmatrix} \ddot{x} + Px = Q\theta \\ \ddot{\theta} + R\theta = \left(\dfrac{Q}{r^2}\right)x \end{bmatrix} \quad ...6.76$$

Almost in all earlier cases, we ended up with two differential equations, one for each mass which are coupled with respect to the two coordinates. In the above cases 'Q' is termed as the coupling coefficient since if $Q = 0$, the two equations are independent or uncoupled of each other and therefore give the two motions, one is rectilinear and the other one is angular. These can exist independently of each other with their respective natural frequencies $\sqrt{P}$ and $\sqrt{R}$ and in case of uncoupled system when $Q = 0$, it means $k_1 l_1 = k_2 l_2$ the natural frequencies in the rectilinear and angular modes respectively, are given as below:

$$\begin{bmatrix} \omega_{n1} = \sqrt{P} = \dfrac{\sqrt{k_1 + k_2}}{m} \\ \omega_{n2} = \sqrt{R} = \sqrt{\dfrac{k_1 l_1^2 + k_2 l_2^2}{J}} \end{bmatrix} \quad ...6.77$$

Now consider the coupled equation 6.76 and let us assume the principal mode of vibration.

Let $x = X \sin \omega t, \quad \ddot{x} = -X\omega^2 \sin \omega t$ and

$$\theta = \beta \sin \omega t, \quad \ddot{\theta} = -\beta\, \omega^2 \sin \omega t \qquad ...6.78a$$

$$\begin{bmatrix} \ddot{x} = -X\omega^2 \sin \omega t \\ \ddot{\theta} = -\beta\omega^2 \sin \omega t \end{bmatrix} \qquad ...6.78b$$

Substutiting these values in equations 6.76 and simplifying, we get

$$\begin{bmatrix} [-\omega^2 + P]\, X = Q\beta \\ [-\omega^2 + R] = \left(\dfrac{Q}{r^2}\right) X \end{bmatrix} \qquad ...6.79$$

By these equations we get the amplitude ratios:

$$\frac{X}{\beta} = \frac{Q}{P - \omega^2} \qquad ...6.80$$

$$\frac{X}{\beta} = \frac{R - \omega^2}{\dfrac{Q}{r^2}} \qquad ...6.81$$

Therefore, $\dfrac{Q}{P - \omega^2} = \dfrac{R - \omega^2}{\dfrac{Q}{r^2}}$

By simplifying these we get the frequency equation as

$$\omega^4 - (P + R)\, \omega^2 + \left\{ PR - \frac{Q^2}{r^2} \right\} = 0 \qquad ...6.82$$

This is in the form of a quadric equation of ω^2, and the roots of the above equation gives the following two natural frequencies of the system.

$$\omega^2_{n1} = \frac{1}{2}(P + R) - \sqrt{\frac{1}{4}(R - P)^2 + \frac{Q^2}{2}}$$

$$\omega^2_{n2} = \frac{1}{2}(P + R) + \sqrt{\frac{1}{4}(R - P)^2 + \frac{Q^2}{r^2}} \qquad ...6.83$$

These two natural frequencies reduce to that of equations 6.77 when $Q = 0$ for the uncoupled case and the mode shape can be got in the usual manner. Also the expression will not be much meaningful in this particular case due to complexity.

EXAMPLE 6.20

An electric motor rotating at 1500 rev/min. drives a centrifugal pump at 500 rev/min, through a single-stage reduction gearing is as shown Fig. 6.4(a). The moments of inertia of the pump impeller and the electric motor are 1400 kg-m^2 and 400 kg-m^2 respectively. The pump shaft and the motor shaft are 45 cm and 18 cm long respectively and their respective diameters are 9 cm and 4.5 cm. Determine the natural frequencies of oscillation. Neglect inertia of gears and $G = 0.8 \times 10^6$ kg/cm^2.

Solution $N_1 = 1500$ rpm, $N_2 = 500$ rpm, $I_1 = 400$ kg-m^2, $I_2 = 1400$ kg-m^2, $l_1 = 18$ cm, $l_2 = 45$ cm, $d_1 = 4.5$ cm, $d_2 = 9$ cm, $G = 0.8 \times 10^6$ kg/cm^2.

Gear ratio $$n = \frac{N_1}{N_2}, \quad n = \frac{1500}{500} = 3$$

By torsional equation, $$\frac{T}{I_P} = \frac{G\theta}{l}, \quad \frac{T}{\theta} = k_t = \frac{GI_P}{l} = \frac{G\pi d^4}{32l}$$

$$\therefore \quad k_{t1} = \frac{0.84 \times 10^{11} \times \pi \times (0.045)^4}{32 \times 0.45}, \quad k_{t1} = 187869.70 \text{ N-m/rad}$$

$$\therefore \quad k_{t2} = \frac{0.84 \times 10^{11} \times \pi \times (0.09)^4}{32 \times 0.45}, \quad k_{t2} = 1202366.05 \text{ N-m/rad}$$

The given system can be reduced as shown in Fig. 6.4(b),

where $$I_e \frac{I_2}{n^2} = \frac{1400}{3^2}, \quad I_e = 155.56 \text{ kg-m}^2$$

$$k_{te} = \frac{k_{t2}}{n^2} = \frac{1202366.05}{3^2}, \quad k_{te} = 133596.23 \text{ N-m/rad}$$

Again the system reduces to Fig. 6.4(c). Then the FBD is as shown in Fig. 6.4(d),

where $$\frac{1}{k_t} = \frac{1}{k_{t1}} + \frac{1}{k_{te}} \qquad \therefore \frac{1}{k_t} = \frac{1}{187869.70} + \frac{1}{133596.23}$$

$\therefore \qquad k_t = 78075.72$ N-m/rad

Applying Newton's second law of motion to the disc (1), $\theta_2 > \theta_1$

$$\Sigma M = I\ddot{\theta} \qquad \therefore\ k_t(\theta_2 - \theta_1) = I_1\ddot{\theta}_1$$

$$I_1\ddot{\theta}_1 + k_t\theta_1 - k_t\theta_2 = 0 \qquad \text{...6.84}$$

This is the differential equation of motion for the disc (1).

Applying Newton's second law of motion to the disc (2),

$$\Sigma M = I\ddot{\theta},\ -k_t(\theta_2 - \theta_1) = I_e\ddot{\theta}_2, \qquad I_e\ddot{\theta}_2 + k_t\theta_2 - k_t\theta_1 = 0 \qquad \text{...6.85}$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using these values in Eq. 6.84,

$$I_1(-A\omega^2) + k_t A = k_t B, \qquad \frac{A}{B} = \frac{k_t}{k_t - I_1\omega^2} \qquad \text{... 6.86}$$

Using the values of $\theta_1\ \theta_2, \ddot{\theta}_2$ in Eq. 6.85,

$$I_e(-B\omega^2) + k_t B = k_t A, \quad \frac{A}{B} = \frac{-I_e\omega^2 + k_t}{k_t} = \frac{k_t - I_e\omega^2}{k_t} \qquad \text{...6.87}$$

From equations 6.86 and 6.87, $\dfrac{k_t}{k_t - I_1\omega^2} = \dfrac{k_t + -I_e\omega^2}{k_t}$, $(k_t - I_1\omega^2)(k_t - I_e\omega^2) = k_t^2$

$$I_1 I_e\omega^4 - [k_t I_1 + k_t I_e]\,\omega^2 + k_t^2 = k_t^2, \quad \omega^2[I_1 I_e\omega^2 - (k_t I_1 + k_t I_e)] = 0$$

$$\omega_{1n}^2 = 0, \quad \omega_{2n}^2 = \frac{k_t I_1 + k_t I_e}{I_1 I_e},$$

$$\omega_{2n}^2 = \frac{78075.72\,[400 + 155.56]}{400 \times 155.56},$$

$$\omega_{2n}^2 = 697.10 \quad \therefore\ \omega_{1n} = 0,\ \omega_{2n} = 26.40 \text{ rad/s}$$

Since one of the natural frequencies of the system is equal to zero, the system is semidefinite.

EXAMPLE 6.21

A spring-mass ($k_1 - M$) system is being acted upon by a harmonic force $F = F_0 \sin \omega t$ (force acting on the mass) as shown in Fig. p-6.21. Another ($k_2 - m$) system is attached to the mass 'M'. Analyse the system to show that the second system may act as a vibration absorber if properly designed. Mention how to design it.

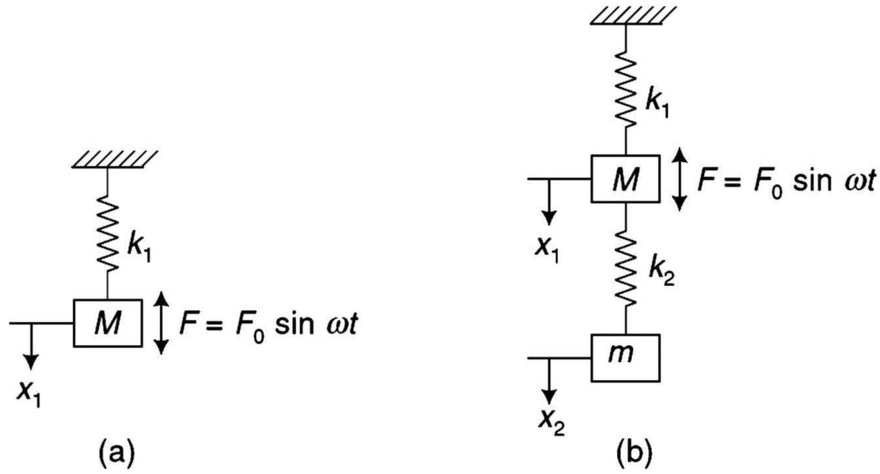


Fig. p-6.21 *Spring-mass system*

Solution As per the statement of the problem, the system is as shown in Fig. p-6.21.

The system is of two degrees of freedom with a forcing function acting on mass 'M'.

Applying Newton's second law of motion to mass 'M' in Fig. p-6.21, the equations of motion are

$$M\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_0 \sin \omega t \qquad ...6.95$$

$$m\ddot{x}_2 + k_2(x_2 - x_1) = 0 \qquad ...6.96$$

This is the differential equation of motion of masses 'M' and 'm'.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t, \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t, \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t, \quad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Substituting these values in equations 6.95 and 6.96,

$$M(-\omega^2 A \sin \omega t) + k_1 A \sin \omega t + k_2(A \sin \omega t - B \sin \omega t) = F_0 \sin \omega t$$

$$m(-\omega^2 B \sin \omega t) + k_2(B \sin \omega t - A \sin \omega t) = 0$$

$$(k_1 + k_2 - M\omega^2) A - k_2 B = F_0, \quad -k_2 A + (k_2 - m\omega^2) B = 0$$

Solving the above equation, $A = \dfrac{F_0(k_2 - m\omega^2)}{(k_1 + k_2 - M\omega^2)\,(k_2 - m\omega^2) - k_2^2}$

In order to cut down the amplitude of vibration of mass 'M', i.e. $A = 0$, $(k_2 - m\omega^2)$ must be equal to zero. Hence, $k_2 = m\omega^2$ or $\omega^2 = \sqrt{\dfrac{k_2}{m}}$

$$\therefore \qquad \omega = \sqrt{\frac{k_2}{m}} \text{ rad/s}$$

The absorber must be, therefore, so designed that its natural frequency is equal to the impressed frequency. When this happens, the amplitude of vibration of mass 'M' is practically zero.

In general, an absorber is used only when the natural frequency of the original system is close to the forcing frequency. Hence, $\dfrac{k_1}{M} = \dfrac{k_2}{m}$ is approximately true for the entire system.

EXAMPLE 6.22

A two-degree-freedom system is as shown in Fig. p-6.22(a).

Determine the amplitude of masses 'M' and 'm'. What modifications are necessary if the sub-system is to act as a dynamic vibration absorber under the following condition?

(i) Spring stiffness kept constant.

(ii) The amplitude of the absorber mass is limited to 0.01 cm.

Solution $M = 10$ kg, $m = 0.5$ kg, $k_1 = 4000$ N/m, $k_2 = 500$ N/m, $F = 50 \cos 21t$

Let us at any instant give a vertical displacement 'x' to the mass 'm_1' as shown in Fig. p-6.22(a). The FBD is as shown in Fig. p-6.22(b).

Applying Newton's second law of motion to mass 'M' $\Sigma F = M\ddot{x}$

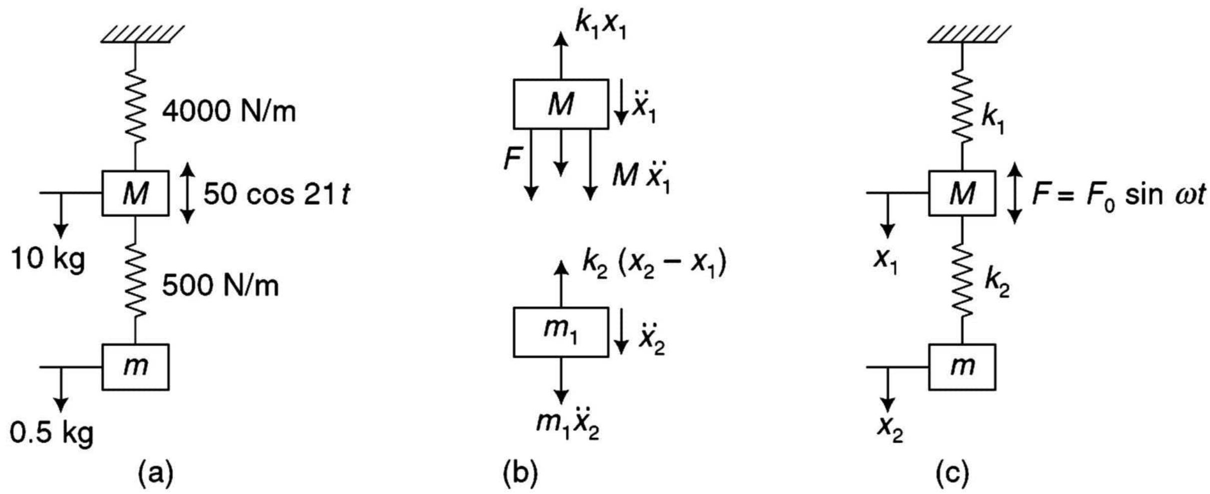


Fig. p-6.22 *Two-degree-freedom system*

$$k_2(x_2 - x_1) + F - k_1x_1 = M\ddot{x}, \quad M\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = F \qquad ...6.97$$

Applying Newton's second law of motion to the mass 'm',

$$\Sigma F = m\ddot{x} \quad -k_2 (x_2 - x_1) = m\ddot{x}_2, \; m\ddot{x}_2 + k_2 x_2 - k_2 x_1 \qquad \text{...6.98}$$

This is the differential equation of motion of masses 'M' and 'm'.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be

$$x_1 = X_1 \cos \omega t, \qquad x_2 = X_2 \cos \omega t$$

$$\ddot{x}_1 = -X_1 \omega^2 \cos \omega t, \quad \ddot{x}_2 = -X_2 \omega^2 \cos \omega t$$

Using these values in equations 6.97 and 6.98,

$$M(-X_1 \omega^2 \cos \omega t) + (k_1 + k_2) X_1 \cos \omega t - k_2 X_2 \cos \omega t = F_0 \cos \omega t$$

$$X_1 [k_1 + k_2 - M\omega^2] - k_2 x_2 = F_0 \qquad \text{...6.99}$$

$$m(-X_2 \omega^2 \cos \omega t) + k_2 X_2 \cos \omega t - k_2 X_1 \cos \omega t = 0$$

$$X_2 [k_2 - m\omega^2] - k_2 X_1 = 0, \; X_2 = \left[\frac{k_2}{k_2 - m\omega^2}\right] X_1 \qquad \text{...6.100}$$

Using Eq. 6.100 in Eq. 6.99, $X_1 = \left[(k_1 + k_2 - M\omega^2) - \dfrac{k_2^2}{(k_2 - m\omega^2)}\right] = F_0$

$$X_1 = \left[\frac{F_0 (k_2 - m\omega^2)}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}\right] \qquad \text{... 6.101}$$

Using Eq. 6.101 in Eq. 6.100,

$$X_2 = \left[\frac{F_0 k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}\right] \qquad \text{...6.102}$$

$\therefore$ amplitude of the mass 'M' is

$$X_1 = \left[\frac{50\,(500 - 0.5 \times 21^2)}{(4000 + 500 - 10 \times 21^2)(500 - 0.5(21)^2) - (500)^2}\right]$$

$$X_1 = -\,6.215 \times 10^{-2}\ \text{m}$$

$\therefore$ amplitude of the mass 'm' is

$$X_2 = \left[\frac{50 \times 500}{(4000 + 500 - 10 \times 21^2)(500 - 0.5(21)^2) - k_2^2}\right]$$

$$X_2 = -\,1.112 \times 10^{-1}\ \text{m}$$

(i) For the sub-system to act as dynamic vibration absorber, $x_1 = 0$

From Eq. 6.101, $F_0 (k_2 - m\,\omega^2) = 0$, $\omega^2 = \dfrac{k_2}{m}$

Keeping k_2 constant, $m = \dfrac{k_2}{\omega^2}$

$\therefore$ mass 'm' $\dfrac{500}{(21)^2}$, $m = 1.13$ kg

(ii) Given $X_2 = 0.001$ m

For dynamic vibration absorber, $X_1 = 0$

i.e. $(k_2 - m\omega^2) = 0$

From Eq. 6.102, $X_2 = \dfrac{-F_0}{k_2}$ $\therefore$ $k_2 = \dfrac{-F_0}{X_2}$

$k_2 = \dfrac{-50}{0.0001} = 5 \times 10^5$ N/m. Negative sign may be neglected.

Note: The vibration absorber must be so designed that its natural frequency is equal to the forcing frequency, i.e. $\omega = \omega_2$, $\omega = \sqrt{\dfrac{k_2}{m_2}}$ rad/s

When this happens, the amplitude of vibration of the mass 'm_1' of the original system is equal to zero. In general, a dynamic vibration absorber is used only when the natural frequency of the original system is close to the forcing frequency (resonance).

Hence, $\omega \approx \omega_1 \approx \omega_2$, or $\dfrac{k_1}{m_1} \approx \dfrac{k_2}{m_2}$ is true for the entire system.

EXAMPLE 6.23

Determine the pitch and bounce frequencies and the location of oscillation centers of an automobile with the following data: m = 1000 kg, r_g = 0.9 m, distance between the front axle and centre of gravity = 1 m, distance between the rear axle and centre of gravity = 1.5 m. Front spring stiffness, k_1 = 8 kN/m, rear spring stiffness k_2 = 22 kN/m.

Solution The equation of motion can be written as

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta),\ m\ddot{x} + (k_1 + k_2)x + (k_2l_2 - k_1l_1)\theta = 0$$

$$\ddot{x} + \left(\frac{k_1 + k_2}{m}\right)x + \left(\frac{k_2l_2 - k_1l_1}{m}\right)\theta = 0 \qquad ...6.114$$

$$J\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2,\ mr_g^2\ddot{\theta} + (k_2l_2 - k_1l_1)x + (k_1l_1^2 + k_2l_2^2)\theta = 0$$

$$\ddot{\theta} + \left(\frac{k_2l_2 - k_1l_1}{mr_g^2}\right)x + \left(\frac{k_1l_1^2 - k_2l_2^2}{mr_g^2}\right)\theta = 0 \qquad ...6.115$$

Substitute the values of m, k, r and l in equations 6.114 and 6.115.

$$\ddot{x} + 30x + 25\theta = 0,\quad \ddot{\theta} + 30.86x + 70.98\theta = 0$$

Assuming the solution as $x = X\sin\omega t$ $\ddot{x} = -\omega^2 X\sin\omega t$

$$\theta = \beta\sin\omega t,\quad \ddot{\theta} = -\omega^2\beta\sin\omega t,\ -\omega^2X + 30X + 25\beta = 0,$$

$$-\omega^2\beta + 30.86X + 70.98\beta = 0$$

$$\frac{X}{\beta} = \frac{25}{\omega^2 - 30} \qquad ...6.116$$

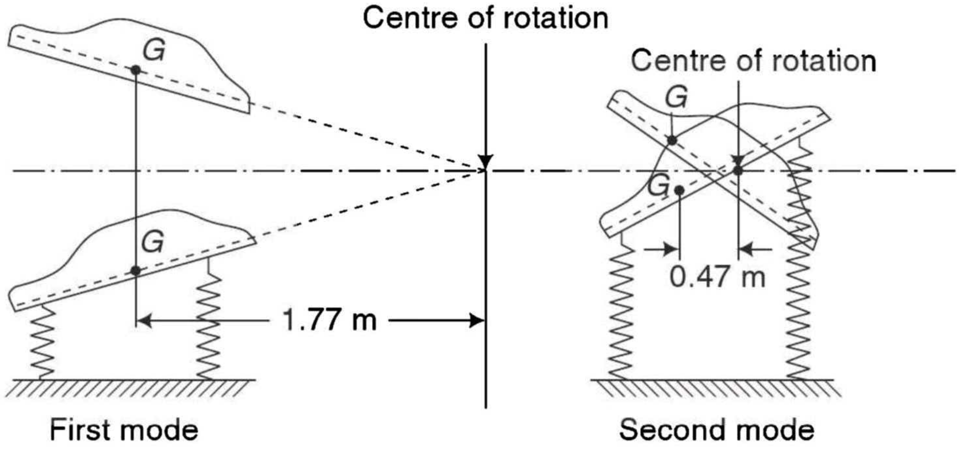


Fig. p-6.23 *Oscillation centers of automobile*

$$\frac{X}{\beta} = \frac{\omega^2 - 70.98}{30.86} \qquad \text{...6.117}$$

Equating equations 6.116 and 6.117, we get $\omega^4 - (30 + 70.98)\omega^2 + 1357.9 = 0$

$$\omega^4 - 100.98\,\omega^2 + 1357.9 = 0$$

$$\omega_{1,2}^2 = \frac{100.98 \pm \sqrt{100.98^2 - 4 \times 1357.9}}{2} \text{ or } \omega_1 = 3.99\text{rad/s}$$

and $\quad \omega_2^2 = 85.0$ or $\omega_2 = 9.24$ rad/s

$$\left(\frac{X}{b}\right)_1 = \frac{25}{\omega_1^2 - 30} = \frac{25}{3.99^2 - 30} = -\frac{1.77}{1} \text{ m/rad}$$

$$\left(\frac{X}{\beta}\right)_2 = \frac{\omega_2^2 - 70.98}{30.86} = \frac{9.24^2 - 70.98}{30.86} = \frac{0.466}{1} \text{ m/rad.}$$

EXAMPEL 6.24

A schematic diagram representation of an automobile is shown in Fig. p-6.24(a) if the automobile weighs 4000 N and has a radius of gyration about the centre of gravity of 4.5 m. The combined spring stiffness of front springs k_1 and k_2 are 3000 N/m and 3250 N/m respectively. Determine the natural frequency of the system.

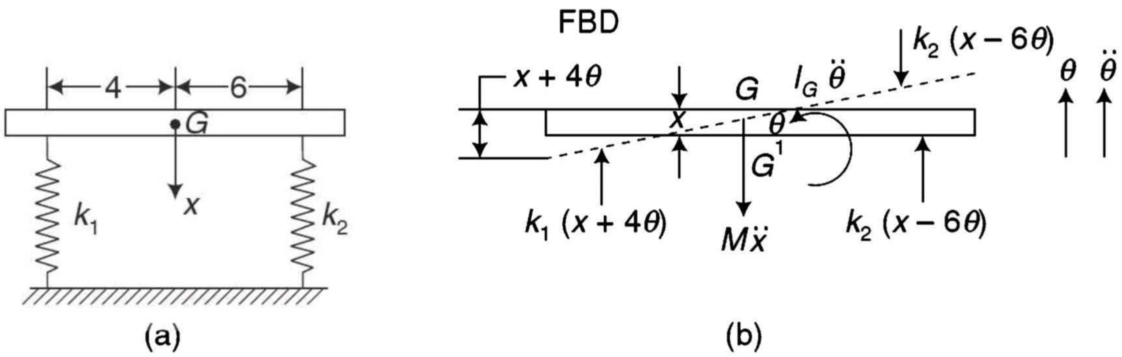


Fig. p-6.24 *Automobile system*

Solution Here, $I_G = MK^2$, K = Radius of gyration

$$I_G = \frac{4000}{9.81} \times (4.5)^2, \quad I_G = 8256.88 \text{ kg} - m^2, \quad M = 407.75 \text{ kg}$$

Considering the linear moment of the mass, apply Newton's second law of motion. The FBD is as shown in Fig. p-6.24(b).

$$-k_1(x + 4\theta) - k_2(x - 6\theta) = M\ddot{x}$$

$$\therefore \quad M\ddot{x} + k_1 x + k_2 x + 4k_1\theta - 6k_2\theta = 0$$

$$M\ddot{x} + (k_1 + k_2)x + (4k_1 - 6k_2)\theta = 0$$

$$407.75\ddot{x} + (3000 + 3250)x + (4 \times 3000 - 6 \times 3250)\theta = 0$$

$$407.75\ddot{x} + 6250x + (-7500)\theta = 0, \quad \ddot{x} + 15.33x - 18.39\theta = 0 \qquad ...6.118$$

Considering the rotation of the mass about its centre of gravity,

$$\Sigma M_G = I_G\ddot{\theta}, \quad -k_1(x + 4\theta)4 + k_2(x - 6\theta)6 = I_G\ddot{\theta}$$

$$I_G\ddot{\theta} + 4k_1(x + 4\theta) - 6k_2(x - 6\theta) = 0, \; I_G\ddot{\theta} + (4k_1 - 6k_2)x - (16k_1 + 36k_2)\theta = 0$$

$$8256.88\,\ddot{\theta} + (4 \times 3000 - 6 \times 3250)x + (16 \times 3000 + 36 \times 3250)\theta = 0$$

$$\ddot{\theta} + 19.98\theta + 0.91x = 0 \qquad ...6.119$$

Put $\quad x = \sin\omega t \qquad \theta = B\sin\omega t$

$$\ddot{x} = -A\omega^2\sin\omega t, \; \ddot{\theta} = -B\omega^2\sin\omega t$$

Using these value of x, θ and $\ddot{\theta}$ in Eq. 6.118,

$$-A\omega^2 + 15.33A - 18.39B = 0, \quad A[15.33 - \omega^2] = 18.39B$$

$$\frac{A}{B} = \frac{18.39}{15.33 - \omega^2} \qquad ...6.120$$

Using the value of x, θ and $\ddot{\theta}$ in Eq. 6.119,

$$-B\omega^2 + 19.98B - 0.91A = 0, \quad B[19.98 - \omega^2] = 0.91A$$

$$\frac{A}{B} = \frac{19.98 - \omega^2}{0.91} \qquad ...6.121$$

From equations 6.120 and 6.121,

$$\frac{18.39}{15.33 - \omega^2} = \frac{19.91 - \omega^2}{0.91}, \; (15.33 - \omega^2)(19.98 - \omega^2) = 18.39 \times 0.91$$

$$15.33 \times 19.98 - (15.33 + 19.98)\omega^2 + \omega^4 = 18.39 \times 0.91, \; \omega^4 - 35.31\,\omega^2 + 289.56 = 0$$

This is the frequency equation which is quadratic in ω^2.

$$\therefore \quad \omega^2 = \frac{35.31 \pm \sqrt{(35.31)^2 - 4 \times 289.56}}{2},$$

$$\omega_{1n}^2 = 12.95 \; \omega_{1n} = 3.6 \text{ rad/s}, \quad \omega_{2n}^2 = 22.36, \quad \omega_{2n} = 4.73 \text{ rad/s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

To draw the principal mode shapes,

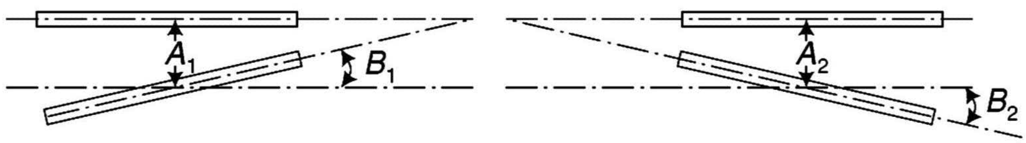
$$\frac{A}{B} = \frac{18.39}{15.33 - \omega^2} \qquad \frac{A}{B} = \frac{18.39}{15.33 - \omega^2}$$

At $\omega^2 = \omega_{1n}^2 = 12.95$ At $\omega^2 = \omega_{2n}^2 = 22.36$

$$\frac{A_1}{B_1} = \frac{18.39}{15.33 - 12.95} \qquad \frac{A_2}{B_2} = \frac{18.39}{15.33 - 22.36}$$

$$\frac{A_1}{B_1} = \frac{18.39}{2.38} \qquad \frac{A_2}{B_2} = \frac{18.39}{-7.03}$$

i.e. $A_1 = 18.39$, $B_1 = 2.38$,   i.e. $A_2 = 18.39$, $B_2 = -7.03$.



Principal modes

EXAMPLE 6.25

Determine the equation of motion of the double pendulum as shown in Fig. p-6.25(a) for small oscillation by using Lagrange's method.

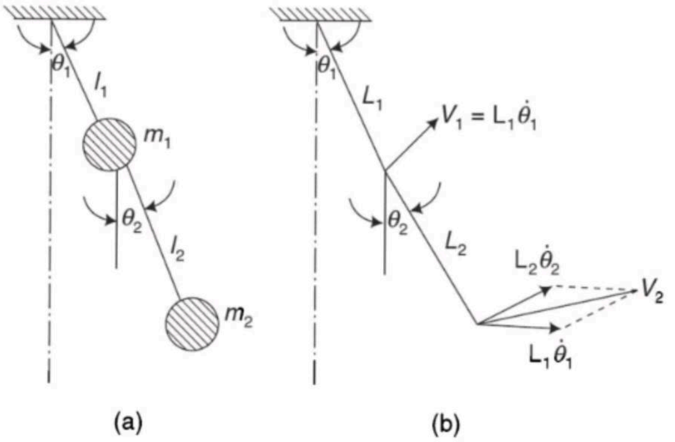


Fig. p-6.25 *Double pendulum*

Solution The KE of the system is KE $= \dfrac{1}{2} m_1 v_1^2 + \dfrac{1}{2} m_2 v_2^2$

$$v_1^2 = (L_1\dot{\theta}_1)^2$$

where $v_2^2 = (L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1L_2\,\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$

Which are velocities of the masses m_1 and m_2 respectively.

$$\text{PE} = m_1 q L_1 (1 - \cos\theta_1) + m_2 g [L_1 (1 - \cos\theta_1) + L_2 (1 - \cos\theta_2)$$

Lagrange's equation is

$$\frac{d}{dt}\frac{\partial(\text{KE})}{\partial q_i} + \frac{\partial(\text{PE})}{\partial q_i} + \frac{\partial(\text{KE})}{\partial q_i} = 0$$

$$\frac{d}{dt}\frac{\partial(\text{KE})}{\partial q_i} = \frac{d}{dt}(m_1L_1^2\,\dot{\theta}_1 + m_2\,[L_1^2\,\dot{\theta}_1 + L_1L_2\,\theta_2\cos(\theta_2 - \theta_1)])$$

$$= m_1L_1^2\,\dot{\theta}_1 + m_2\left[L_1^2\,\dot{\theta}_1 + L_1L_2\,\dot{\theta}_2\cos(\theta_2 - \theta_1) + L_1L_2\,\dot{\theta}_2\frac{d}{dt}(m_2\cos(\theta_2 - \theta_1))\right]$$

$$= mL_1^2\,\ddot{\theta}_1 + m_2L_1^2\,\ddot{\theta}_1 + m_2L_1L_2\,\dot{\theta}_2$$

where $\sin\theta = \theta$, $\cos(\theta_2 - \theta_1) = 1$ and $\dfrac{d}{dt}[\cos(\theta_2 - \theta_1)] = 0$ since θ is small.

Also $\dfrac{\partial(\text{KE})}{\partial\theta_i} = 0.\quad \dfrac{\partial(\text{PE})}{\partial\theta_i} = m_1gL_1\sin\theta_1 - m_2gL_1\sin\theta_1$

Then the first equation of motion is given by

$$(m_1 + m_2)\,L_1\,\ddot{\theta}_1 + m_1L_2\,\ddot{\theta}_2 + (m_1 + m_2)\,g\theta_1 = 0$$

Similarly, $\dfrac{d}{dt}\dfrac{\partial(\text{KE})}{\partial\theta_2} = \dfrac{d}{dt}[m_2L_2^2\,\ddot{\theta}_2 + m_2L_1L_2\,\dot{\theta}_1\cos(\theta_2 - \theta_1)]$

$$= m_2L_2^2\,\ddot{\theta}_2 + m_2L_1L_2\,\dot{\theta}_1$$

$$\frac{\partial(\text{KE})}{\partial\theta_2} = 0$$

$$\frac{\partial(\text{PE})}{\partial\theta_2} = m_2gL_2\sin\theta_2$$

And so the second equation of motion becomes

$$L_2\,\ddot{\theta}_2 + g\theta_2 + L_1\,\ddot{\theta}_1.$$

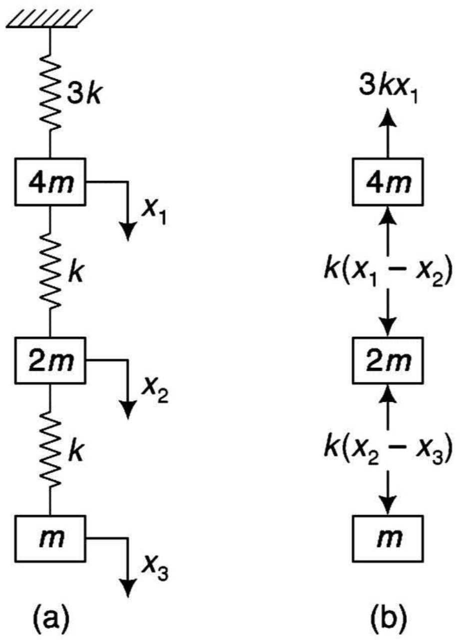
EXAMPLE 7.1

Determine the natural frequencies of the three-degree-freedom spring-mass (linear) system by using Newton's method as shown in Fig. p-7.1.

Solution Now at any instant give vertical displacement 'x_1'to the mass '$4m$', 'x_2' to the mass '$2m$'and 'x_3' to the mass 'm' as shown in Fig. p-7.1(a). The FBD is as shown in Fig. p-7.1(b) assuming that $x_1 > x_2 > x_3$.

Then the two lower springs are in compression and the top spring is in tension for the direction of x_1 as shown in Fig. p-7.1(b). Then the various spring forces acting are as shown in FBD of Fig. p-7.1(b).

Now applying Newton's second law of motion, $\Sigma F = m\ddot{x}$, the equations of motion are



Multi-degree linear spring-mass system

$$4m\ddot{x}_1 = -3kx_1 - k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 = k(x_1 - x_2) - k(x_2 - x_3) = 0,\ m\ddot{x}_3 = k(x_2 - x_3) = 0$$

$$4m\ddot{x}_1 + 3kx_1 + k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Rearranging the above equations,

$$\left.\begin{aligned} 4m\ddot{x}_1 + 4kx_1 - kx_2 &= 0 \\ 2m\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 &= 0 \\ m\ddot{x}_3 + kx_3 - kx_2 &= 0 \end{aligned}\right\} \quad ...7.100$$

This is the differential equation of motion of the masses 'm_1', 'm_2'and 'm_3'.

For solution of equations 7.100, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = X_1 \sin \omega t,\ \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$$

$$x_2 = X_2 \sin \omega t,\ \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$

$$x_3 = X_3 \sin \omega t,\ \ddot{x}_3 = -\omega^2 X_3 \sin \omega t$$

Substituting these values in equations 7.100

$\because \ \sin \omega t \neq 0$

$$\left.\begin{aligned} (4k - 4m\omega^2) X_1 - kX_2 &= 0 \\ (2k - 2m\omega^2) X_2 - kX_1 - kX_3 &= 0 \\ (k - m\omega^2) X_3 - kX_2 &= 0 \end{aligned}\right\} \quad ...7.101$$

To find the natural frequency equation, the determinant of the coefficient of x_1, x_2, and x_3 must be equated to zero.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 4(k - m\omega^2) & -k & 0 \\ -k & 2(k - m\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Expand the determinant to get the frequency equations:

$$4(k - m\omega^2)\lfloor 2(k - m\omega^2)(k - m\omega^2) - k^2 \rfloor + k\lfloor -k(k - m\omega^2) - 0 \rfloor + 0 = 0$$

$$(k - m\omega^2)\{\lfloor 8(k - m\omega^2)^2 - 4k^2 \rfloor - k^2\} = 0$$

$$(k - m\omega^2)\lfloor 8k^2 + 8m^2\omega^4 - 16\,km\omega^2 - 5k^2 \rfloor = 0$$

$$(k - m\omega^2)\lfloor 8m^2\omega^4 - 16km\omega^2 - 3k^2 \rfloor = 0$$

$$(k - m\omega^2) = 0,\ 8m^2\omega^4 - 16km\omega^2 - 3k^2 = 0$$

$$k = m\omega^2 \quad \omega^2 = \frac{k}{m} \quad \omega = \sqrt{\frac{k}{m}} \text{ rad/s}$$

$$\omega_{a,b}^2 = \frac{16km \pm \sqrt{(16km)^2 - 4(8m^2)(3k^2)}}{2 \times 8m^2} = \frac{16km \pm \sqrt{256k^2m^2 - 96k^2m^2}}{16m^2}$$

$$\omega^2_{a,b} = \frac{16km \pm \sqrt{256k^2m^2 - 96k^2m^2}}{16m^2} = \frac{16km \pm 12.65km}{16m^2}, = \frac{k}{m} \pm \frac{12.65}{16}\frac{k}{m}$$

$$\omega_a^2 = 0.2094\frac{k}{m} \qquad \omega_b^2 = 1.79\frac{k}{m}$$

$$\omega_a = 0.4576\sqrt{\frac{k}{m}} \text{ rad/s} \qquad \omega_b = 1.338\sqrt{\frac{k}{m}} \text{ rad/s}$$

Hence, the natural frequencies are

$$\omega_{n1} = 0.46\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n2} = \sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n3} = 1.34\sqrt{\frac{k}{m}} \text{ rad/s.}$$

EXAMPLE 7.2

Determine the natural frequencies of the three-degree-freedom spring-mass (linear) system by using Newton's method as shown in Fig. p-7.2(a).

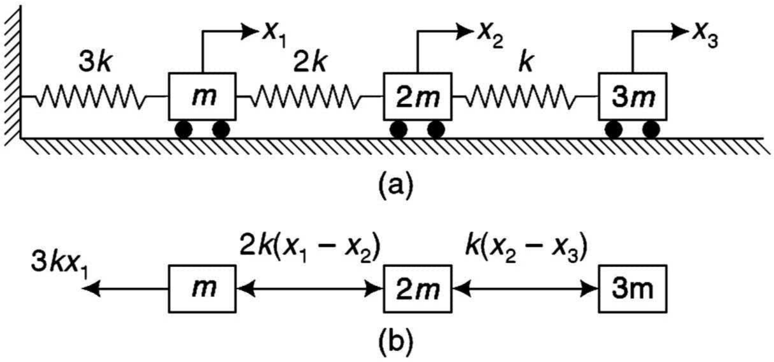


Fig. p-7.2 *Multi-degree linear spring-mass system*

Solution Now at any instant give linear displacement 'x_1' to the mass 'm', 'x_2' to the mass '$2m$' and 'x_3' to the mass '$3m$' as shown in Fig. p-7.2(a). The FBD is as shown in Fig. p-7.2(b) assuming that $x_1 > x_2 > x_3$.

Then the other two springs ($2k$ and k) are in compression and the top spring is in tension for the direction of x_1 as shown in Fig. p-7.2(b).Then the various spring forces acting are as shown in FBD of Fig. p-7.2(b).

Now applying Newton's second law of motion, $\Sigma F = m\ddot{x}$, the equations of motion are

$$m\ddot{x}_1 + 3kx_1 + 2k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + 2k(x_2 - x_1) + k(x_2 - x_3) = 0 \ 3m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Rearranging the above equations,

$$\left.\begin{aligned} m\ddot{x}_1 + 5kx_1 - 2kx_2 &= 0 \\ 2m\ddot{x}_2 + 3kx_2 - 2kx_1 - kx_3 &= 0 \\ 3m\ddot{x}_3 + kx_3 - kx_2 &= 0 \end{aligned}\right\} \qquad \ldots 7.102$$

This is the differential equation of motion of the masses 'm_1', 'm_2' and 'm_3'.

For solutions of equations 7.102, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies.

Let one of these components be,

$$x_1 = X_1 \sin \omega t, \quad \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$$
$$x_2 = X_2 \sin \omega t, \quad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$
$$x_3 = X_3 \sin \omega t, \quad \ddot{x}_3 = -\omega^2 X_3 \sin \omega t$$

Substituting these values in equations 7.102

$$-m\omega^2 X_1 + 5kX_1 - 2kX_2 = 0 \qquad \because \sin \omega t \neq 0$$
$$-2m\omega^2 X_2 + 3kX_2 - 2kX_1 - kX_3 = 0$$
$$-3m\omega^2 X_3 + kX_3 - kX_2 = 0$$

$$\left.\begin{aligned} (5k - m\omega^2)\, X_1 - 2kX_2 &= 0 \\ (3k - 2m\omega^2)\, X_2 - 2kX_1 - kX_3 &= 0 \\ (k - 3m\omega^2)\, X_3 - kX_2 &= 0 \end{aligned}\right\} \qquad \text{...7.103}$$

To find the natural frequency equation, the determinant of the coefficients of x_1, x_2 and x_3 must be equated to zero.

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{vmatrix} (5k - m\omega^2) & -2k & 0 \\ -2k & (3k - 2m\omega^2) & -k \\ 0 & -k & (k - 3m\omega^2) \end{vmatrix} = 0$$

Expand the determinant to get the frequency equation

$$(5k - m\omega^2)\lfloor(3k - 2\, m\omega^2)\,(k - 3m\omega^2) - k^2\rfloor + 2k\lfloor -2k\,(k - 3m\omega^2)\rfloor = 0$$
$$(5k - m\omega^2)\,[2k^2 - 9\, mk\omega^2 - 2mk\omega^2 + 6m^2\omega^4] + 2k[-2k^2 + 6mk\omega^2] = 0$$
$$\omega^6 - 6.83\,\frac{k}{m}\,\omega^4 + 7.5\,\frac{k^2}{m^2}\,\omega^2 - \frac{k^3}{m^3} = 0$$

By solving the above equation, the natural frequencies are

$$\omega_{n1} = 0.396\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n2} = 1.084\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n3} = 2.35\sqrt{\frac{k}{m}} \text{ rad/s.}$$

EXAMPLE 7.3

Determine the natural frequencies of the three-rotor (semi-definite) system by using Newton's method as shown in Fig. p-7.3.

Solution Now at any instant give angular displacement 'θ_1'to the disc'J_1', 'θ_2' to the disc 'J_2'and 'θ_3' to the disc 'J_3' as shown in Fig. p-7.3(a). The FBD is as shown in Fig. p-7.3(b) assuming that $\theta_1 > \theta_2 > \theta_3$.

Now applying Newton's second law of motion, $J\ddot{\theta} = -\Sigma T$, the equations of motion are

$$J_1\,\ddot{\theta}_1 = -k_t(\theta_1 - \theta_2)$$
$$J_2\ddot{\theta}_2 = k_t\,(\theta_1 - \theta_2) - k_t\,(\theta_2 - \theta_3)$$
$$J_3\ddot{\theta}_3 = -k_t\,(\theta_2 - \theta_3)$$

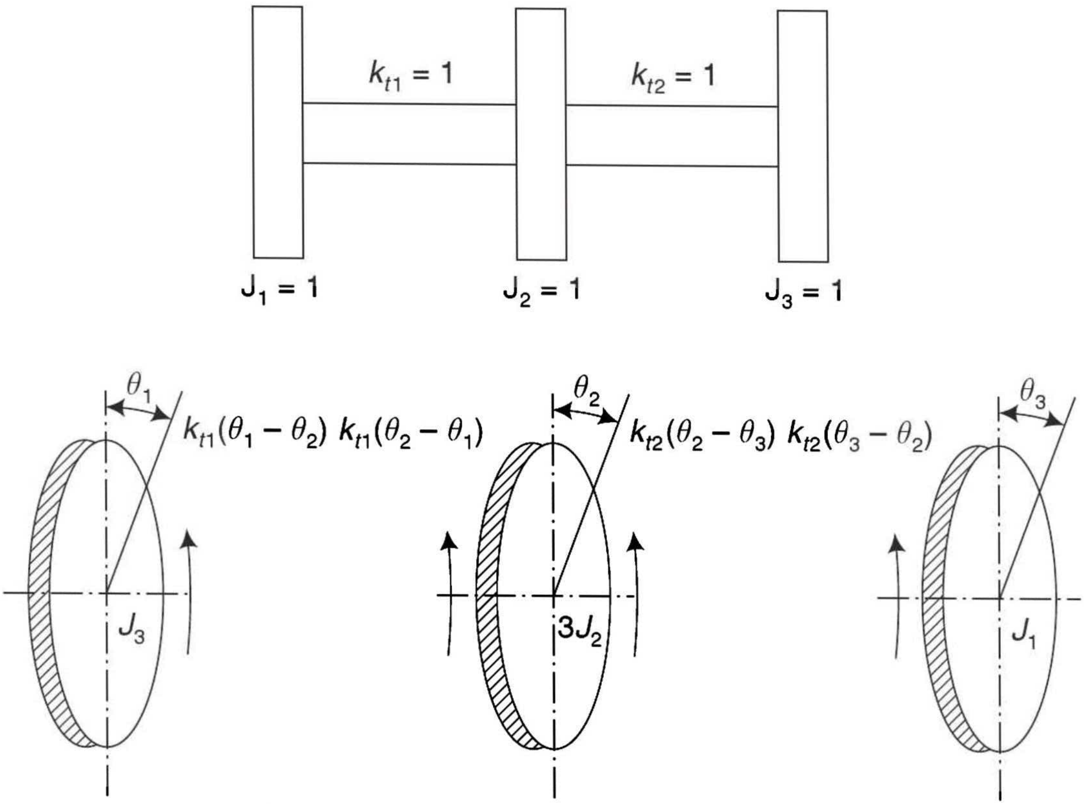


Fig. p-7.3 *Multi-degree torsional system*

$$\left.\begin{aligned} J_1\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) &= 0 \\ J_2\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) &= 0 \\ J_3\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) &= 0 \end{aligned}\right\} \qquad \text{...7.104}$$

This is the differential equation of motion of the discs 'J_1', 'J_2'and 'J_3'.

For solutions of equations 7.104, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be

$$\begin{aligned} \theta_1 &= a \sin \omega t, & \ddot{\theta}_1 &= -\omega^2 a \sin \omega t \\ \theta_2 &= b \sin \omega t, & \ddot{\theta}_2 &= -\omega^2 b \sin \omega t \\ \theta_3 &= c \sin \omega t, & \ddot{\theta}_3 &= -\omega^2 c \sin \omega t \end{aligned}$$

Substituting these values in equations 7.104,

$$\begin{aligned} &(k_{t1} - J_1\omega^2)\, a - k_{t1} b = 0 \\ &(k_{t1} + k_{t2} - 3J_2\omega^2)\, b - 2k_{t1} a - k_{t2} c = 0 \\ &(k_{t2} - J_3\omega^2)\, c - k_{t2} b = 0 \end{aligned}$$

To find the natural frequency equation, the determinant of the coefficient of a, b and c must be equated to zero.

$$\begin{vmatrix} \theta_1 & \theta_2 & \theta_3 \\ (k_{t1} - J_1\omega^2) & -k_{t1} & 0 \\ -k_{t1} & (k_{t1} + k_{t2} - J_2\omega^2) & -k_{t2} \\ 0 & -k_{t2} & (k_{t2} - J_3\omega^2) \end{vmatrix} = 0$$

$$(k_{t1} - J_1\omega^2)\,[(k_{t1} + k_{t2} - J_2\omega^2)\,(k_{t2} - J_3\omega^2) - k_{t2}^2] + k_{t1}[-\,k_{t1}\,(k_{t2} - J_3\omega^2) - 0] + 0 = 0$$

By simplying the above equation, we get

$$\omega^6 - \left[\frac{k_{t1}}{J_1} + \frac{k_{t1}}{J_2} + \frac{k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right]\omega^4 + \left[\frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1\,J_2\,J_3}\right]\omega^2 = 0$$

$$\omega^2\,[\omega^4 - \left\{\frac{k_{t1}}{J_1} + \frac{k_{t1} + k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right\}\omega^2 + \left\{\frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1J_2J_3}\right\} = 0$$

$\omega_1^2 = 0$ ($\because$ semidefinite system).

$$\omega_{2,3}^2 = +\left\{\frac{k_{t1}}{2J_1} + \frac{k_{t1} + k_{t2}}{2J_2} + \frac{k_{t2}}{2J_3}\right\} \pm \sqrt{\left\{\frac{k_{t1}}{2J_1} + \frac{k_{t1} + k_{t2}}{2J_2} + \frac{k_{t2}}{2J_3}\right\}^2 - \frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1J_2J_3}}$$

By solving the above equation, the natural frequencies are

$$\omega^2{}_{n1} = 0 \text{ rad/s},\ \omega_{n2} = \sqrt{\frac{k_{t2}}{J_2}} \text{ rad/s},\ \omega_{n3} = 1.74\sqrt{\frac{k_{t2}}{J_3}} \text{ rad/s.}$$

EXAMPLE 7.4

Find the influence coefficients of the spring-mass system as shown in Fig. p-7.4.

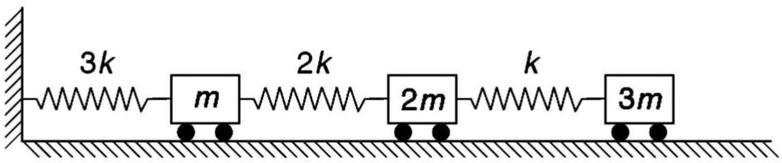


Fig. p-7.4 *Spring-mass system*

Solution Apply unit load at position '1'of Fig. p-7.4.

The influence coefficient, $a_{11} = \dfrac{1}{3k}$ (deflection at 1 due to unit load at 1)

$a_{21} = \dfrac{1}{3k}$ (deflection at 2 due to unit load at 1), $a_{31} = \dfrac{1}{3k}$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{ij} = a_{ji}$,

$$\therefore \qquad a_{21} = a_{12} = \frac{1}{3k}$$

$$\therefore \qquad a_{31} = a_{13} = \frac{1}{3k}$$

Apply unit load at position '2' of Fig. p-7.4.

Neglecting the mass at 1, springs '3k' and 'k' are in series.

$$\therefore \qquad a_{22} = \frac{1}{k_{eq}}$$

where $\qquad \dfrac{1}{k_{eq}} = \dfrac{1}{3k} + \dfrac{1}{2k}$ or $a_{22} = \dfrac{5}{6k}$ (deflection at 2 due to load at 2)

$$a_{32} = \frac{5}{6k} \text{ (deflection at 3 due to load at 2),}$$

By Maxwell's reciprocal theorem, $a_{32} = a_{23} = \dfrac{5}{6k}$

Apply unit load at position '3' of Fig. p-7.4.

Neglecting the masses at points '1' and '2', springs '$3k$', '$2k$' and 'k' are in series

$$\therefore \qquad \frac{1}{k_{eq}} = \frac{1}{3k} + \frac{1}{2k} + \frac{1}{k} = \frac{11}{6k},$$

$$\therefore \qquad a_{33} = \frac{1}{k_{eq}}$$

$$\therefore \qquad a_{33} = \frac{11}{6k} \text{ (deflection at 3 due to unit load at 3).}$$

EXAMPLE 7.5

Find the influence coefficient of the system as shown in Fig. p-7.5(a) and thus find the values of natural frequencies.

Solution Apply unit load at the position 1.

Considering mass $m_1 = m$ in Fig. p-7.5(b), $T \sin \theta = 1$, $T \cos \theta = 3mg$,

$$\therefore \qquad \tan \theta = \frac{1}{3mg}$$

For small angles of 'θ' $\tan \theta \approx \sin \theta = \dfrac{1}{3mg}$

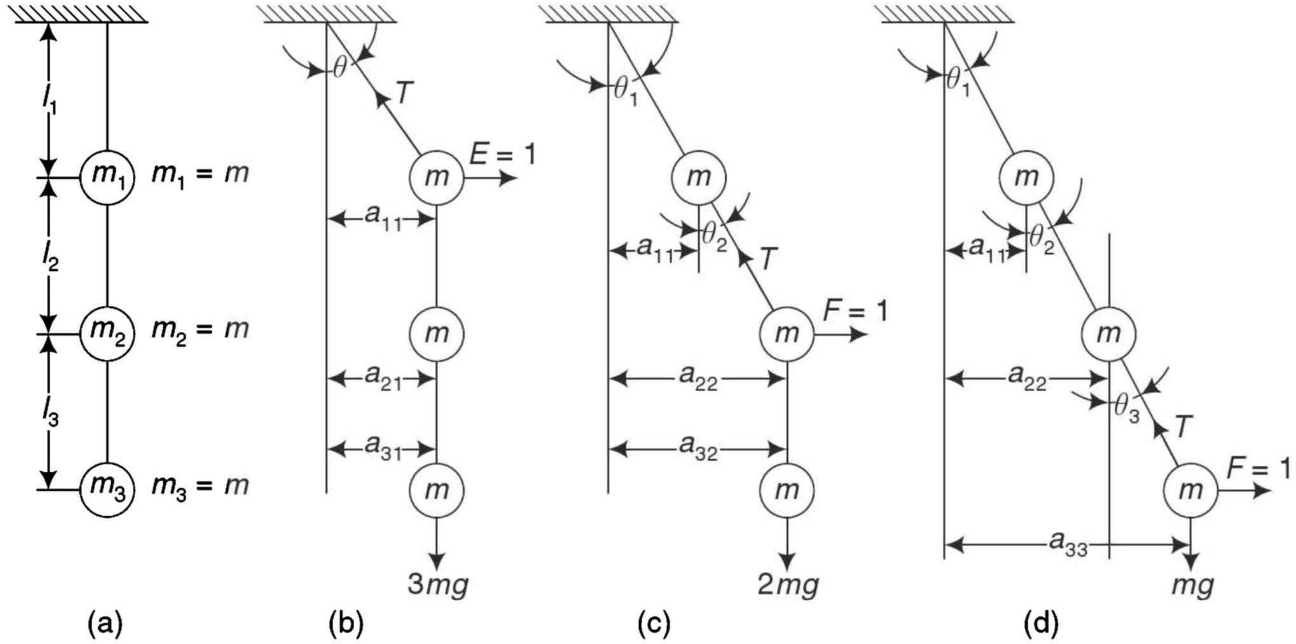


Fig. p-7.5 *System of multiple masses*

From the geometry of Fig. p-7.5(b), $\sin \theta = \dfrac{a_{11}}{l}$

$$\therefore \qquad a_{11} = l \sin \theta$$

$$a_{11} = \frac{1}{3mg} = a_{21} = a_{31} = a_{12} = a_{13}$$

Apply unit load at the position 2.

Considering mass $m_2 = m$, $\Sigma V = 0$ and $\Sigma H = 0$

$$\therefore \qquad T \sin \theta_1 = 1, T \cos \theta_1 = 2\,mg$$

$$\therefore \qquad \tan \theta_1 = \frac{1}{2mg}$$

For small angles of θ_1 $\tan \theta_1 \approx \sin \theta_1 = \dfrac{1}{2mg}$

From the geometry of Fig. p-7.5(c), $a_{22} = a_{11} + l \sin \theta_1$

$$a_{22} = \frac{1}{3mg} + \frac{1}{2mg}, a_{22} = \frac{5l}{6mg} = a_{32} = a_{23}$$

Apply unit load at the position 3.

Considering mass $m_3 = m$, $\Sigma V = 0$ and $\Sigma H = 0$

$$\therefore \qquad T \sin \theta_2 = 1, T \cos \theta_2 = mg \quad \therefore \tan \theta_2 = \frac{1}{mg}$$

For small angles of θ_2, $\tan \theta_2 \approx \sin \theta_2 = \dfrac{1}{mg}$

From the geometry of Fig. p-7.5(d),

$$a_{33} = a_{22} + l \sin \theta^2, a_{33} = \frac{5l}{6mg} + \frac{1}{mg}, = \frac{11l}{6mg}$$

The equation of motion using influence coefficient in matrix form is as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} m_1 a_{11} & m_2 a_{12} & m_3 a_{13} \\ m_1 a_{21} & m_2 a_{22} & m_3 a_{23} \\ m_1 a_{31} & m_2 a_{32} & m_3 a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} \frac{ml}{3mg} & \frac{ml}{3mg} & \frac{ml}{3mg} \\ \frac{ml}{3mg} & \frac{5ml}{6mg} & \frac{5ml}{6mg} \\ \frac{ml}{3mg} & \frac{5ml}{6mg} & \frac{11ml}{6mg} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{ml\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find the first principal mode and first natural frequency,

let $\qquad x_1 = x_2 = x_3 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \frac{l\omega^2}{g} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$B_2 = 0A_2 + B_2 + 0C_2$$
$$C_2 = 0A_2 + 0B_2 + C_2$$

This can be written in matrix form as follows:

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -2.29 & -3.92 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} \text{ (Sweeping matrix)}$$

To obtain the second natural frequency, the sweeping matrix is combined with the matrix of the first principal mode.

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 0 & -2.29 & -3.92 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \frac{l\omega^2}{6g}$$

For first iteration, let $x_1 = 1, x_2 = 1, x_3 = 1.$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -8.42 \\ -2.42 \\ 3.58 \end{bmatrix} = \frac{8.42\, l\omega^2}{6g} \begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix}$$

For second iteration, let $x_1 = -1,\ x_2 = 0.29, x_3 = 0.43$

$$\begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -1.76 \\ -1.34 \\ 1.24 \end{bmatrix} = \frac{1.76\, l\omega^2}{6g} \begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix}$$

For third iteration, $x_1 = -1, x_2 = -0.76, x_3 = 0.70$

$$\begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix}$$

For second iteration, $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 12 \\ 27 \\ 45 \end{bmatrix} = \frac{12 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix}$$

For third iteration, $x_1 = 1,\ x_2 = 2.25,\ x_3 = 3.75$

$$\begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14 \\ 32 \\ 54.70 \end{bmatrix} = \frac{14 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix}$$

For fourth iteration, $x_1 = 1,\ x_2 = 2.29,\ x_3 = 3.89$

$$\begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14.36 \\ 32.90 \\ 56.24 \end{bmatrix} = \frac{14.36 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix}$$

For fifth iteration, $x_1 = 1,\ x_2 = 2.29,\ x_3 = 3.92$

$$\begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14.42 \\ 33.04 \\ 56.57 \end{bmatrix} = \frac{14.42 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix}$$

Since the assumed values is approximately equal to the obtained values, the first principal modes will be, $A_1 = 1$, $B_1 = 2.29$, $C_1 = 3.92$.

The first natural frequency is

$$\frac{14.42 l\omega^2}{6g} = 1, \text{ or } \omega_{1n}^2 = \frac{6}{14.92} \cdot \frac{g}{l} \text{ or } \quad \therefore \ \omega_{1n} = 0.65\sqrt{\frac{g}{l}} \text{ rad/s.}$$

To obtain the second principal modes, the orthogonality principle is used.

$\therefore \ m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$, or $m\,(1)\,A_2 + m\,(2.29)\,B_2 + m\,(3.92)\,C_2 = 0$

$$A_2 = -2.29,\ B_2 = -3.92\ C_2,\ A_2 = 0A_2 - 2.29B_2 - 3.92C_2$$

$$= \frac{l\omega^2}{6g} = \begin{bmatrix} -2.13 \\ -2.31 \\ 1.89 \end{bmatrix} = \frac{2.13 l\omega^2}{6g} = \begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix}$$

For fourth iteration, $x_1 = -1$, $x_2 = -1.08$, $x_3 = 0.89$

$$\begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -2.41 \\ -2.98 \\ 2.36 \end{bmatrix} = \frac{2.41 l\omega^2}{6g} \begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix}$$

For fifth iteration, $x_1 = -1$, $x_2 = -1.24$, $x_3 = 0.98$

$$\begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -2.52 \\ -3.30 \\ 2.58 \end{bmatrix} = \frac{2.52 l\omega^2}{6g} \begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix}$$

For sixth iteration, $x_1 = -1$, $x_2 = -1.31$, $x_3 = 1.02$

$$\begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -2.58 \\ -3.45 \\ 2.67 \end{bmatrix} = \frac{2.58 l\omega^2}{6g} \begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix}$$

For seventh iteration, $x_1 = -1$, $x_2 = -1.34$, $x_3 = 1.04$

$$\begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -2.62 \\ -3.52 \\ 2.72 \end{bmatrix} = \frac{2.62 l\omega^2}{6g} \begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix}$$

Since the assumed values and the obtained values are approximately equal, the second principal modes are given by $A_2 = -1$, $B_2 = -1.34$, $C_2 = 1.04$

$\therefore$ second natural frequency is given by

$$\frac{2.62 l\omega^2}{6g} = 1 \text{ or } \omega_{2n}^2 = \frac{6g}{2.62 l} \quad \therefore \ \omega_{2n} = 1.51\sqrt{\frac{g}{l}} \text{ rad/s}$$

To obtain the third natural frequency and third principal modes, the orthogonality principle should be used.

$$\therefore \quad m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

Using the values,

$$m(-1)\,A_3 + m\,(1.34)\,B_3 + m\,(1.04)\,C_3 = 0,$$

$$-A_3 - 1.34\,B_3 + 1.04\,C_3 = 0 \qquad \text{...7.105}$$

$$m\,(1)\,A_3 + m\,(2.29)\,B_3 + m\,(3.92)\,C_3 = 0$$

$$-A_3 + 2.29\,B_3 + 3.92\,C_3 = 0 \qquad \text{...7.106}$$

Adding equations 7.105 and 7.106, we get

$$0.95B_3 + 4.96C_3 = 0$$

$$B_3 = -5.22C_3 \qquad \text{...7.107}$$

Substituting the value of B_3 in Eq. 7.105, we get

$$-A_3 - 1.34\,(-5.22)\,C_3 + 1.04C_3 = 0$$

$$A_3 = 8.03\,C_3 \qquad \text{...7.108}$$

Writing the equation in terms of C_3 from equations 7.107 and 7.108,

$$A_3 = 0A_3 + 0B_3 + 8.03\,C_3$$

$$B_3 = 0A_3 + 0B_3 - 5.22\,C_3$$

$$C_3 = 0A_3 + 0B_3 + C_3$$

The sweeping matrix will become

$$\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8.03 \\ 0 & 0 & -5.22 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}$$

When this sweeping matrix is combined with the matrix equation of the second mode, we get the matrix of third mode.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} 0 & 0 & 8.03 \\ 0 & 0 & -5.22 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For first iteration, $x_1 = 1, x_2 = 1, x_3 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 7.63 \\ -5.03 \\ 0.97 \end{bmatrix} = \frac{7.63 l\omega^2}{6g} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix}$$

For second iteration, $x_1 = 1, x_{2,} = -0.66, x_{3,} = 0.13$

$$\begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0.99 \\ -0.65 \\ 0.13 \end{bmatrix} = \frac{0.99 l\omega^2}{6g} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix}$$

Since the assumed value is approximately equal to the obtained value, the third principal modes will be, $A_3 = 1$, $B_3 = -0.66$, $C_3 = 0.13$

The third natural frequency will be $0.99l\omega^2/6g = 1$, $\omega^2_{3n} = 6g/0.99l$,

$$\therefore \qquad \omega_{3n} = 2.46\sqrt{\frac{g}{l}} \text{ rad/s.}$$

EXAMPLE 7.6

Determine the influence coefficient of the triple pendulum of lengths 'l_1','l_2' and 'l_3' and masses 'm_1', 'm_2' and 'm_3' as shown in Fig. p-7.6(a).

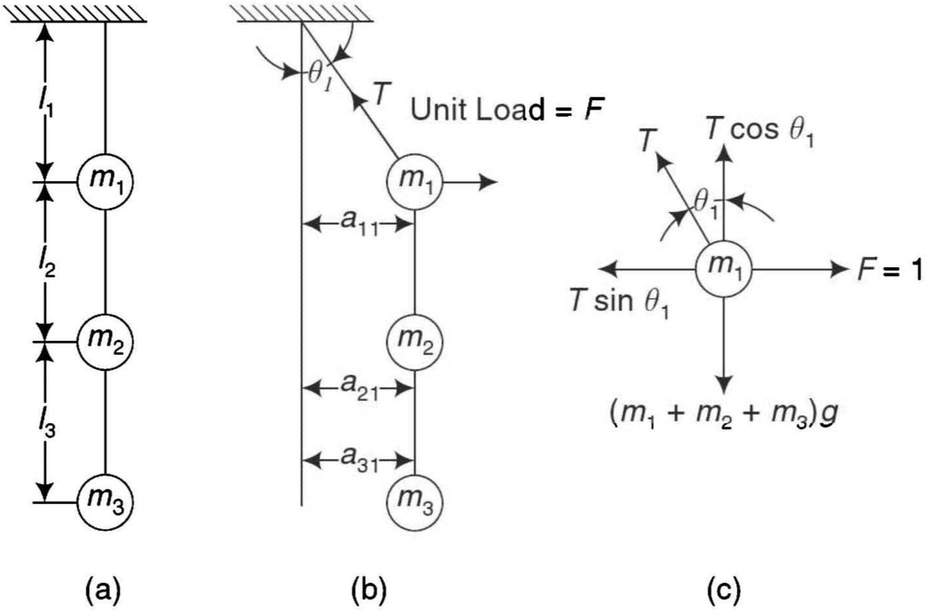


Fig. p-7.6 *Triple pendulum*

Solution Apply unit load to mass 'm_1'.

Considering mass 'm_1',

$$\Sigma V = 0 \text{ and } \Sigma H = 0$$

$$\therefore \qquad T \sin \theta_1 = 1 \qquad \text{...7.109}$$

$$\therefore \qquad T \cos \theta_1 = (m_1 + m_2 + m_3)g \qquad \text{...7.110}$$

Divide Eq. 7.109 by Eq. 7.110, $\tan \theta_l = \dfrac{1}{(m_1 + m_2 + m_3)g}$

For small angles of θ_1, $\tan \theta_1 \cong \sin \theta_1$

From the geometry of Fig. p-7.6(b), $\sin \theta_1 = \dfrac{a_{11}}{l_1}$, $a_{11} = l_1 \sin \theta_1$

The influence coefficient, $a_{11} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$

From the geometry of Fig. p-7.6(c), $a_{21} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$, $a_{31} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$

By Maxwell's reciprocal theorem, $a_{21} = a_{12}$ and $a_{31} = a_{13}$

Applying unit load to mass 'm_2', neglecting mass 'm_1',

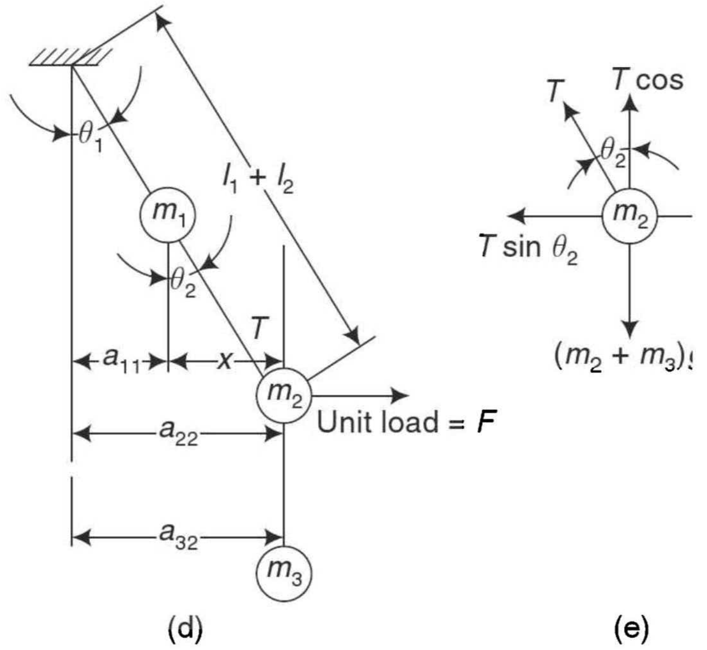


Fig. p-7.6 *Contd.*

Considering mass 'm_2', Fig. p-7.6(d), ΣV and $\Sigma H = 0$

$\therefore \quad T \sin \theta_2 = 1$

$\therefore \quad T \cos \theta_2 = (m_2 + m_3)g$

$\therefore \quad \tan \theta_2 = \dfrac{1}{(m_2 + m_3)g}$

For small angles of θ_2, $\tan \theta_2 \cong \sin \theta_2$

From the geometry of Fig. p-7.6(e), $\sin \theta_2 = \dfrac{x}{l_2}$, $x = l_2 \sin \theta$, $x = \dfrac{l_2}{(m_2 + m_3)g}$

But influence coefficient,

$$a_{22} = a_{11} + x, \; a_{22} = \frac{l_1}{(m_1 + m_2 + m_3)g} + \frac{l_2}{(m_2 + m_3)g}$$

From the geometry of the figure, $a_{32} = a_{22} = \dfrac{l_1}{(m_1 + m_2 + m_3)g} + \dfrac{l_2}{(m_2 + m_3)g}$

Applying unit load at the position '3', neglecting masses 'm_1' and 'm_2'.

Consider mass m_3, Fig. p-7.6(f) $\Sigma V = 0$ and $\Sigma H = 0$

$\therefore \quad T \sin \theta_3 = 1$

$\therefore \quad T \cos \theta_3 = m_3 g$

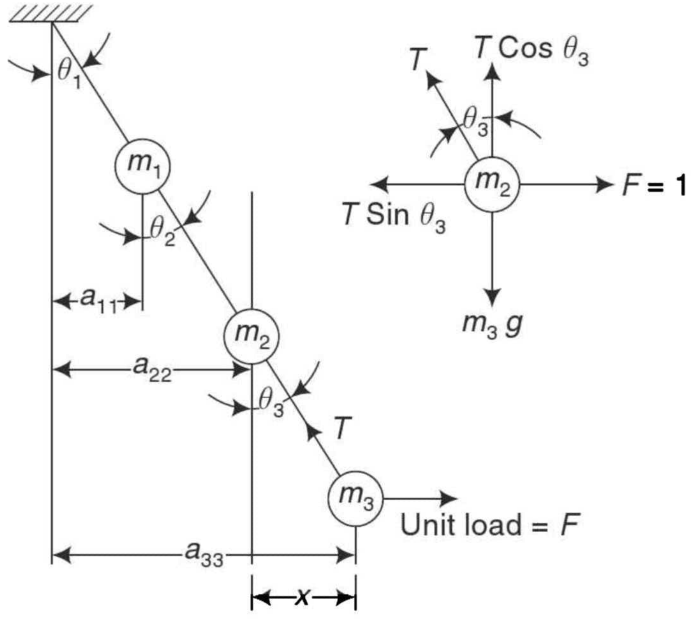
$\therefore \quad \tan \theta_3 = \dfrac{1}{m_3 g}$

For small angle of θ_3, $\tan \theta_3 \cong \sin \theta_3$

From the geometry of Fig. p-7.6(g)

$$\sin \theta_3 = \frac{x}{l_3}, \; x = l_3 \sin \theta_3, \; x = \frac{l_3}{m_3 g}$$

But $\quad a_{33} = a_{22} + x$



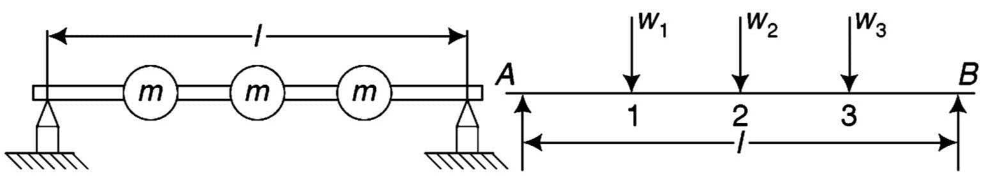
(f) (g)

Fig. p-7.6 *Contd.*

$$a_{33} = \frac{l_1}{(m_1 + m_2 + m_3)g} + \frac{l_2}{(m_2 + m_3)g} + \frac{l_3}{m_3 g}.$$

EXAMPLE 7.7

A simply supported beam of length 'l' has three equal masses attached to it at equal distances as shown in Fig. p-7.7(a). Determine the influence coefficient.



(a) (b)

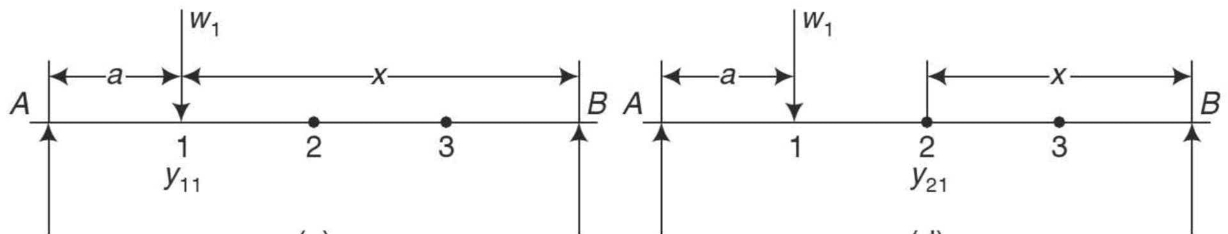
Fig. p-7.7 *Simply supported beam*

Note: Deflection at any point 'x' is given by a simply supported beam.

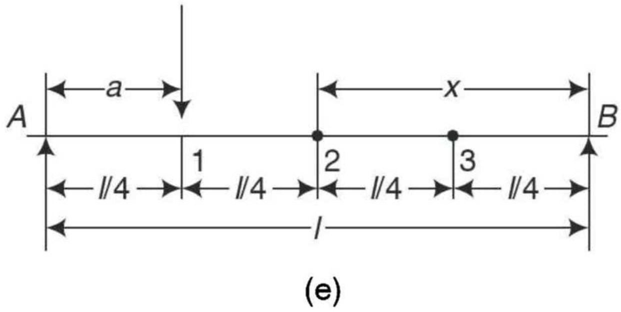
$$y_x = \frac{wax(l^2 - a^2 - x^2)}{6EIl} \quad \text{for } x \le (1 - a)$$

where w = **Load applied at a distance 'a' from the end A or B**

x = **Distance to the point from end B or A, where the deflection is actually required**



(c) (d)



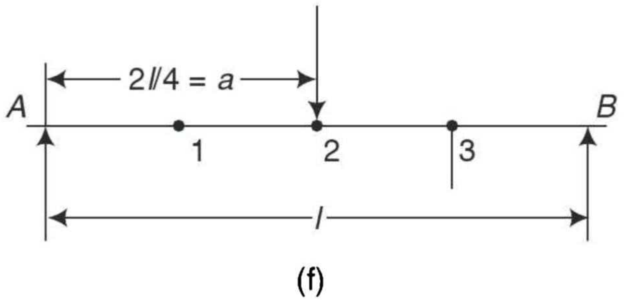


Fig. p-7.7 *Contd.*

E = Young's modulus of the beam material and I = Moment of inertia of the beam

Applying unit load at the point 1, Fig. p-7.7(c).

Solution Influence coefficient, $a = \dfrac{l}{4},\ x = \dfrac{3l}{4}\quad x \le 1 - a$

i.e. $$\frac{3l}{4} \le l - \frac{l}{4} \text{ and } a_{11} = \frac{\dfrac{l}{4} \times \dfrac{3l}{4}\left(l_2 - \left(\dfrac{l}{l_4}\right)^2 - \left(\dfrac{3l}{4}\right)^2\right.}{6EIl}$$

$\therefore$ the condition is satisfied, $$a_{11} = \frac{\dfrac{3l}{16}\left(l_2 - \dfrac{l^2}{16} - \dfrac{9l^2}{16}\right)}{6EI} = \frac{3l^3 \times 6}{16 \times 16 \times 6EI} = \frac{3l^3}{256EI}$$

Deflection at the position '1' due to unit load at position '1', $a = \dfrac{l}{4},\ x = \dfrac{2l}{4}$

Condition $x \le l - a$, i.e. $\dfrac{2l}{4} \le \left(l - \dfrac{l}{4}\right), \dfrac{2l}{4} \le \dfrac{3l}{4}$ (true)

$$\therefore \quad a_{21} = \frac{l \times \dfrac{l}{4} \times \dfrac{2l}{4}\left(l_2 - \left(\dfrac{l}{4}\right)^2 - \left(\dfrac{2l}{4}\right)^2\right)}{6EI} = \frac{\dfrac{2l}{16}\left(l^2 - \dfrac{l^2}{16} - \dfrac{4l^2}{16}\right)}{6EI}$$

$$a_{21} = \frac{2 \times 11l^3}{256EI}\ \frac{l}{6} = \frac{3.67l^3}{256EI}$$ (deflection at 2 due to unit load at 1)

$$a = \frac{l}{4},\ x = \frac{1}{4}$$

Condition $x \le l - a$, i.e. $\dfrac{l}{4} \le \left(l - \dfrac{l}{4}\right) \Rightarrow \dfrac{l}{4} \le \dfrac{3l}{4}$ (True)

$$\therefore \quad a_{31} = \frac{\dfrac{l}{4} \times \dfrac{l}{4}\left(l^2 - \left(\dfrac{l}{4}\right)^2 - \left(\dfrac{l}{4}\right)^2\right)}{6EI} = \frac{14}{6} \times \frac{l^3}{256EI}$$

$$a_{31} = \frac{2.33l^3}{256EI}$$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{21} = a_{12} = \dfrac{3.67l^3}{256EI}\ a_{31} = a_{13} = \dfrac{2.33l^3}{256EI}$

Applying unit load at the point '2', influence coefficient a_{22}, $a = \frac{2l}{4}, x = \frac{2l}{4}$, condition $x \le (l - a)$,

i.e. $$\frac{2l}{4} \le \left(l - \frac{2l}{4}\right) \Rightarrow \frac{2l}{4} \le \frac{2l}{4} \text{ (True)}$$

$$\therefore a_{22} = \frac{\frac{2l}{4}\frac{2l}{4}\left[l^2 - \left(\frac{2l}{4}\right)^2 - \left(\frac{2l}{4}\right)^2\right]}{6EIl,} = \frac{4l(16l^2 - 4l^2 - 4l^2)}{6x256EI} = \frac{5.33l^3}{256EI}$$

(deflection at 2 due to unit load at 2)

$$a = \frac{2l}{4}, x = \frac{l}{4} \text{ condition } x \le (l - a), \text{ i.e. } \frac{l}{4} \le \frac{2l}{4} \text{ (True)}$$

$$\therefore \quad a_{32} = \frac{\frac{2l}{4} \times \frac{l}{4}\left[l^2 - \left(\frac{2l}{4}\right)^2 - \left(\frac{l}{4}\right)^2\right]}{6EIl} = \frac{2l[16l^2 - 4l^2 - l^2]}{6 \times 256EI} = \frac{3.67l^3}{256EI}$$

(deflection at 3 due to load at 2)

Applying unit load at point 3, $\quad a = \frac{1}{4}, x = \frac{3l}{4}$

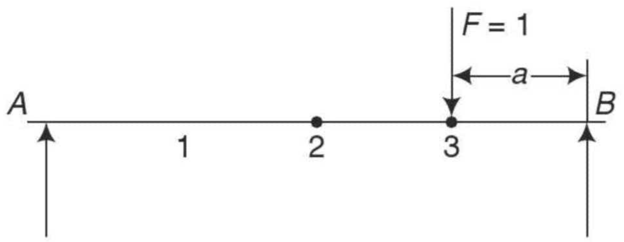


Fig. p-7.7(g) *Contd.*

Condition $\quad x \le (l - a), \quad \therefore \frac{3l}{4} \le \left(l - \frac{l}{4}\right)$

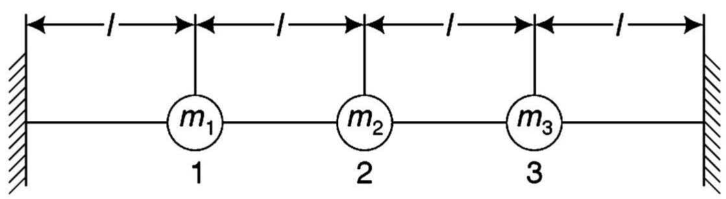
$$\therefore \quad \frac{3l}{4} \le \frac{3l}{4} \text{ (True)}$$

$$\therefore a_{33} = \frac{\frac{l}{4} \times \frac{3l}{4}\left(l^2 - \left(\frac{l}{4}\right)^2 - \left(\frac{3l}{4}\right)^2\right)}{6EIl} = \frac{3l^3}{256EI} \text{ (deflection at 3 due to unit load at 3)}$$

By Maxwell's reciprocal theorem, $a_{32} = a_{23} = \frac{3.67l^3}{256EI}$.

EXAMPLE 7.8

Determine the influence coefficient of a dynamic system consisting of three equal masses attached to a taut string as shown in Fig. p-7.8(a).



(a)

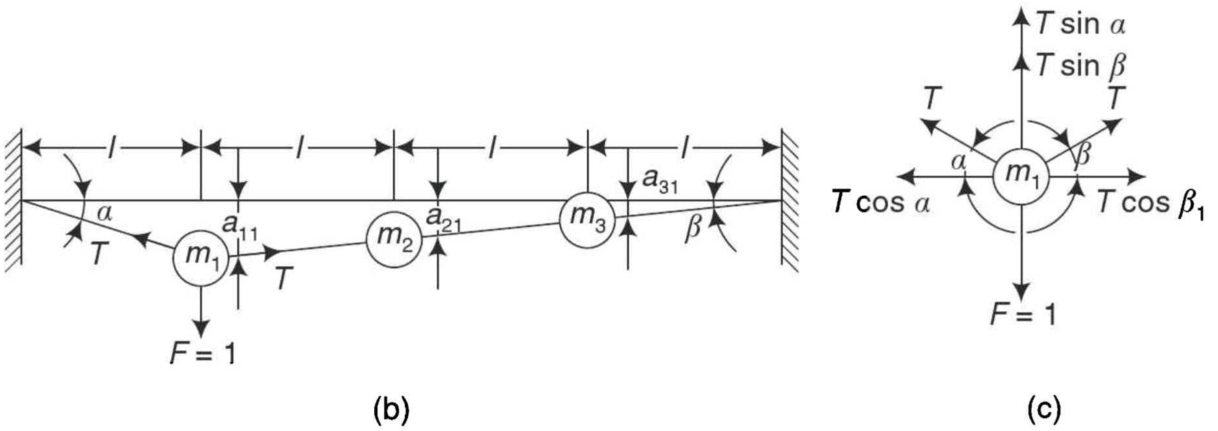


Fig. p-7.8 *Dynamic system*

Solution Applying unit load at the point 1, let 'T' be the tension in the string. For small angles of 'α' and 'β', $\tan\alpha \approx \sin\alpha$, $\tan\beta \approx \sin\beta$

Considering the mass 'm_1' in Fig. 7.8(b) $\Sigma H = 0$

$$\therefore \quad T\cos\alpha = T\cos\beta$$

Considering vertical movement of the mass m_1,

$$\Sigma V = 0$$

$$\therefore \quad T\sin\alpha + T\sin\beta = 1$$

But from the geometry of Fig. 7.8(c), $\sin\alpha = \dfrac{a_{11}}{l}$, $\sin\beta = \dfrac{a_{11}}{3l}$

$T\left(\dfrac{a_{11}}{l} + \dfrac{a_{11}}{3l}\right) = 1$, $a_{11} = \dfrac{3l}{4T}$ (deflection at 1 due to unit load at the point '1')

Comparing similar triangles,

$a_{21} = \dfrac{2}{3} a_{11} = \dfrac{2}{3} \times \dfrac{3}{4T}$ $\quad \therefore a_{21} = \dfrac{l}{2T}$ (deflection at 2 due to unit load at 1)

Comparing similar triangles,

$\dfrac{a_{11}}{3l} = \dfrac{a_{31}}{l}$, $a_{31} = \dfrac{1}{3} a_{11}$, $a_{31} = \dfrac{l}{4T}$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{12} = a_{21} = \dfrac{l}{2T}$ (deflection at 1 due to unit load at 2)

$a_{31} = a_{13} = \dfrac{l}{4T}$ (deflection at 1 due to unit load at 3) applying unit load at the point 2.

Considering the vertical movement of mass 'm_2' in Fig. p-7.8(d)

$$\Sigma V = 0,\ T\sin\theta + T\sin\theta = 1$$

From the geometry of Fig. p-7.8(e),

$$\sin\theta = \frac{a_{22}}{2l}$$

$$\therefore \quad T\left(\frac{l}{2l}\right) + \left(\frac{l}{2l}\right) a_{22} = 1$$

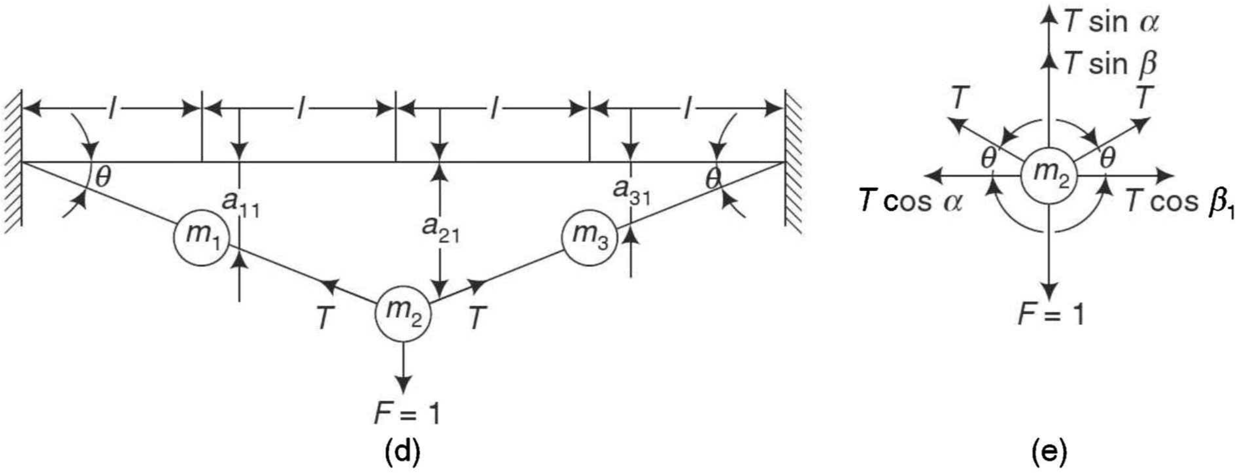


Fig. p-7.8 *Contd.*

$$\therefore \qquad a_{22} = \frac{2l}{2T} = \frac{l}{T}$$

Comparing the similar triangles,

$$\frac{a_{22}}{2l} = \frac{a_{32}}{l}, \; a_{32} = \frac{a_{22}}{2}, \; a_{32} = \frac{l}{2T} \text{ (deflection at 3 due to unit load at 2)}$$

By Maxwell's reciprocal theorem,

$$a_{23} = a_{32} = \frac{l}{2T} \text{ (deflection at 2 due to unit load at 2)}$$

Applying unit load point '3' by symmetry, $a_{33} = a_{11} = \dfrac{3l}{4T}$

Note: For the system shown in Fig. p-7.8.(f)

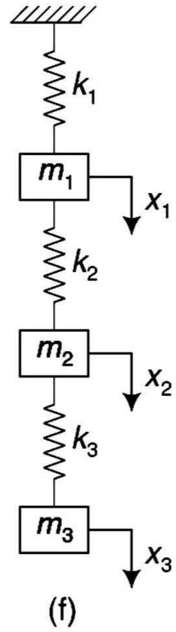


Fig. p-7.8 *Contd.*

$m_1\ddot{x}_1$ is the inertia force of the mass m_1

$m_2\ddot{x}_2$ is the inertia force of the mass m_2

$m_3\ddot{x}_3$ is the inertia force of the mass m_3

Let $\qquad x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t, \quad x_3 = C \sin \omega t$

$$\ddot{x}_1 = -\omega^2 x_1, \; \ddot{x}_2 = -B\omega^2 \sin \omega t, \; \ddot{x}_3 = -\omega^2 x_2$$

$\therefore$ the inertia forces will be $-m_1\omega^2 x_1$, $-m_2\omega^2 x_2$ and $-m_3\omega^2 x_3$

For unit load, influence coefficient = a_{ij}

For inertia force, influence coefficient = (inertia force) a_{ij}

For a three-degree-freedom system shown in Fig. p-7.8.(f),

there are nine influence coefficients:

a_{11}, a_{12}, a_{13} for the mass m_1, a_{21}, a_{22}, a_{23} for the mass m_2, a_{31}, a_{32}, a_{33} for the mass m_3.

$\therefore$ total deflection of masses 'm_1', 'm_2' and 'm_3' is given by

$$\left.\begin{aligned} a_1 &= a_{11} + a_{12} + a_{13} \\ a_2 &= a_{21} + a_{22} + a_{23} \\ a_3 &= a_{31} + a_{32} + a_{33} \end{aligned}\right\} \text{For unit force}$$

Considering inertia forces, displacements are given by

$$-x_1 = a_{11}m_1\ddot{x}_1 + a_{12}m_2\ddot{x}_2 + a_{13}m_3\ddot{x}_3 \quad \therefore\ x_1 = a_{11}m_1\omega^2 x_1 + a_{12}m_2\omega^2 x_2 + a_{13}m_3\omega^2 x_3$$

$$\therefore\ x_2 = a_{21}m_1\omega^2 x_1 + a_{22}m_2\omega^2 x_2 + a_{23}m_3\omega^2 x_3$$

$$\therefore\ x_3 = a_{31}m_1\omega^2 x_1 + a_{32}m_2\omega^2 x_2 + a_{33}m_3\omega^2 x_3$$

This can be written in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} a_{11}m_1\omega^2 x_1 & a_{12}m_2\omega^2 x_2 & a_{13}m_3\omega^2 x_3 \\ a_{21}m_1\omega^2 x_1 & a_{22}m_2\omega^2 x_2 & a_{23}m_3\omega^2 x_3 \\ a_{31}m_1\omega^2 x_1 & a_{32}m_2\omega^2 x_2 & a_{33}m_3\omega^2 x_3 \end{bmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \omega^2 \begin{bmatrix} a_{11}m_1x_1 & a_{12}m_2x_2 & a_{13}m_3x_3 \\ a_{21}m_1x_1 & a_{22}m_2x_2 & a_{23}m_3x_3 \\ a_{31}m_1x_1 & a_{32}m_2x_2 & a_{33}m_3x_3 \end{bmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

EXAMPLE 7.9

Calculate the natural frequencies of the system as shown in Fig. p-7.9 by using matrix method.

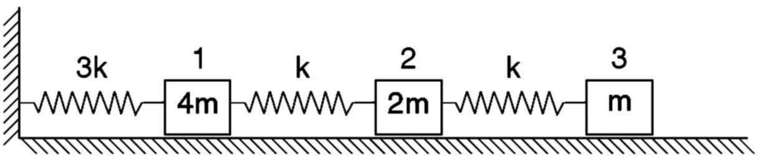


Fig. p-7.9 *Spring-mass system*

To calculate influence coefficient, applying unit load to the position ‘1’

$$a_{11} = \frac{1}{3k} = a_{21} = a_{31} = a_{12} = a_{13}$$

Applying unit load at the position ‘2’, neglect mass at ‘1’,

$$a_{22} = \frac{1}{3k} + \frac{1}{k} = \frac{4}{3k} = a_{32} = a_{23}$$

Applying unit load at the position 3, neglecting masses at ‘2’ and ‘3’,

$$a_{33} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = \frac{7}{3k} \quad \text{given } m_1 = 4m,\ m_2 = 2m,\ m_3 = m$$

The equation of motion for a three-degree-freedom system in matrix form is written as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} m_1a_{11} & m_2a_{12} & m_3a_{13} \\ m_1a_{21} & m_2a_{22} & m_3a_{23} \\ m_1a_{31} & m_2a_{32} & m_3a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} \frac{4m}{3k} & \frac{2m}{3k} & \frac{m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{4m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{7m}{3k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assuming $x_1 = 1, x_2 = 1, x_3 = 1$ for the first iteration,

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4+2+1\\4+8+4\\4+8+7\end{bmatrix}\begin{bmatrix}7\\16\\19\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}7\\16\\19\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{7m\omega^2}{3k}\begin{bmatrix}1\\2.29\\2.71\end{bmatrix}$$

For second iteration, $x_1 = 1, x_2 = 2.29, x_3 = 2.71$

$$\begin{bmatrix}1\\2.29\\2.71\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\2.29\\2.71\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}11.29\\33.16\\41.29\end{bmatrix} = \frac{11.29m\omega^2}{3k}\begin{bmatrix}1\\2.29\\3.66\end{bmatrix}$$

For third iteration, $x_1 = 1, x_2 = 2.94, x_3 = 3.66$

$$\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix} = \frac{11.29m\omega^2}{3k}\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}13.54\\42.16\\53.14\end{bmatrix}$$

$$\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{13.54m\omega^2}{3k}\begin{bmatrix}1\\3.11\\3.92\end{bmatrix}$$

For fourth iteration, $x_1 = 1, x_2 = 3.11, x_3 = 3.92$

$$\begin{bmatrix}1\\3.11\\3.92\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.11\\3.92\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.14\\44.56\\56.32\end{bmatrix} = \frac{14.14m\omega^2}{3k}\begin{bmatrix}1\\3.15\\3.98\end{bmatrix}$$

For fifth iteration, $x_1 = 1, x_2 = 3.15, x_3 = 3.98$

$$\begin{bmatrix}1\\3.15\\3.98\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.15\\3.98\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.28\\45.12\\57.06\end{bmatrix} = \frac{14.28\,m\omega^2}{3k}\begin{bmatrix}1\\3.16\\4\end{bmatrix}$$

For sixth iteration, $x_1 = 1, x_2 = 3.16, x_3 = 4$

$$\begin{bmatrix}1\\3.16\\4\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.16\\4\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.32\\45.28\\57.28\end{bmatrix} = \frac{14.32\,m\omega^2}{3k}\begin{bmatrix}1\\3.16\\4\end{bmatrix}$$

Since the assumed value is very close to the obtained value

$$\frac{14.32\,m\omega^2}{3k} = 1$$

$\therefore\ \omega_{1n}^2 = 0.21\dfrac{k}{m}\ \therefore\ \omega_{1n} = 0.46\sqrt{\dfrac{k}{m}}$ rad/s (first natural frequency)

The first principal modes are given by $A_1 = 1, B_1 = 3.16$ and $C_1 = 4.0$

To obtain the second principal mode, the orthogonality principle is used,

i.e. $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$

$\therefore$ $4m(1)A_2 + 2m(3.16) + B_2 + m(4)C_2 = 0$

or $4A_2 + 6.32B_2 + 4C_2 = 0,\ A_2 = -1.58B_2 - C_2$ or $A_2 = 0A_2 - 1.58B_2 - C_2$

$B_2 = B_2,\ \ B_2 = 0A_2 + B_2 + 0C_2,\ \ C_2 = C_2,\ \ C_2 = 0A_2 + 0B_2 + C_2$

These can be written in matrix form as

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$$

This matrix, if combined with the matrix of first mode, is called sweeping matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Starting the first iteration, let the second principal modes be

$$x_1 = 1,\quad x_2 = 1,\quad x_3 = 1$$

$$\therefore \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -7.32 \\ 1.68 \\ 1.68 \end{bmatrix} = \frac{7.32m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix}$$

For second iteration, $x_1 = -1,\quad x_2 = 0.23,\quad x_3 = 0.64$

$$\begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.91 \\ 0.39 \\ 2.31 \end{bmatrix} = \frac{2.91m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix}$$

For third iteration, $x_1 = -1,\quad x_2 = 0.13,\quad x_3 = 0.79$

$$\begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.93 \\ 0.22 \\ 2.59 \end{bmatrix} = \frac{2.93\,m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix}$$

For fourth iteration, $x_1 = -1,\quad x_2 = 0.08,\quad x_3 = 0.88$

$$\begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.93 \\ 0.13 \\ 2.77 \end{bmatrix} = \frac{2.99\,m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix}$$

For fifth iteration, $x_1 = -1,\quad x_2 = 0.04,\quad x_3 = 0.93$

$$\begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.96 \\ 0.07 \\ 2.86 \end{bmatrix} = \frac{2.96m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix}$$

For sixth iteration, $x_1 = -1, \quad x_2 = 0.02, \quad x_3 = 0.97$

$$\begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -3 \\ 0.03 \\ 2.94 \end{bmatrix} = \frac{3m\omega^2}{3k} - \begin{bmatrix} 1 \\ 0.01 \\ 0.98 \end{bmatrix}$$

For seventh iteration, $x_1 = -1, \quad x_2 = 0.01, \quad x_3 = 0.98$

$$\begin{bmatrix} -1 \\ 0.01 \\ 0.98 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0.01 \\ 0.98 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -2.98 \\ 0.02 \\ 2.96 \end{bmatrix} = \frac{2.98m\omega^2}{3k}\begin{bmatrix} -1 \\ 0.01 \\ 0.99 \end{bmatrix}$$

Since the values of $x_{1=-1}, \quad x_{2=}0, \quad x_3 = 1$

Let for eighth iteration, $x_1 = -1, \quad x_2 = 0, \quad x_3 = 1$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = \frac{3m\omega^2}{3k}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since the obtained modes is equal to the assumed modes,

$$= \frac{3m\omega^2}{3k} = 1, \quad \omega_{2n}^2 = \frac{k}{m}, \quad \omega_{2n} = \sqrt{\frac{k}{m}} \text{ rad/s. (second natural frequency)}$$

To obtain third principal modes, the orthogonality principle is

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

But $\quad A_1 = 1, \; B_1 = 3.16, \quad C_1 = 4, \quad A_2 = -1, \; B_2 = 0, \quad C_2 = 1$

$\therefore \quad 4m(-1)A_3 + 2m(0)B_3 + m(1)C_3 = 0$

$$-4A_3 + 0\,B_3 + C_3 = 0 \qquad \text{...7.111}$$

$$4m(1)A_3 + 2m(3.16)B_3 + m(4)C_3 = 0$$

$$4A_3 + 6.32B_3 + 4C_3 = 0 \qquad \text{...7.112}$$

Solving equations 7.111 and 7.112 and adding, we get

$$6.32\,B_3 \; + 5C_3, \quad B_3 = \frac{-5}{6.32}C_3, \quad B_3 = -0.79\,C_3$$

Or $\quad A_3 = 0A_3 + 0B_3 + 0.25C_3, \; B_3 = 0A_3 + 0B_3 - 0.79C_3, \; C_3 = 0A_3 + 0B_3 + 1C_3$

Writing in matrix form,

$$\left.\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}\right\} \text{(sweeping matrix)}$$

This matrix is combined with the matrix of the second mode.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & 0 & 0.41 \\ 0 & 0 & -1.33 \\ 0 & 0 & 1.67 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For first iteration, $x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 1.0$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0 & 0 & 0.41\\0 & 0 & -1.33\\0 & 0 & 1.67\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0.41\\-1.33\\1.67\end{bmatrix} = \frac{0.41m\omega^2}{3k}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix}$$

For second iteration, $x_1 = 1.0, \quad x_2 = -3.24, \quad x_3 = 4.07$

$$\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0 & 0 & 0.41\\0 & 0 & -1.33\\0 & 0 & 1.67\end{bmatrix}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}1.67\\-5.41\\6.80\end{bmatrix} = \frac{0.41m\omega^2}{3k}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix}$$

Since the assumed amplitudes is equal to the obtained values,

$$\therefore \qquad \frac{1.67m\omega^2}{3k} = 1, \quad \omega^2 = \frac{3}{1.67}\frac{k}{m}, \quad \omega_{3n}^2 = 1.80\frac{k}{m}$$

$$\therefore \qquad \omega_{3n} = 1.34\sqrt{\frac{k}{m}} \text{ rad/s (third natural frequency)}$$

Third principal mode will be $A_3 = 1, B_3 = -3.24, C_3 = 4.07$.

EXAMPLE 7.10

Determine the natural frequencies and principal modes of vibration for the 3-degree-freedom system as shown in Fig. p-7.10 by using matrix iteration method.

Solution Determine the influence coefficient system shown in Fig. p-7.10.

We know that earlier.

$$[\alpha_{ij}] = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\\alpha_{21} & \alpha_{22} & \alpha_{23}\\\alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix} = \begin{bmatrix}\frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k}\\\frac{1}{3k} & \frac{4}{3k} & \frac{4}{3k}\\\frac{1}{3k} & \frac{4}{3k} & \frac{7}{3k}\end{bmatrix}$$

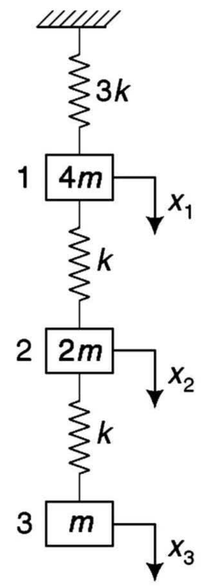


Fig. p-7.10 *Three-degree freedom system*

In the next step, write down the equation of motion using influence coefficients.

$$\alpha_{11}m_1\ddot{x}_1 + \alpha_{12}m_2\ddot{x}_2 + \alpha_{13}m_3\ddot{x}_3 + x_1 = 0$$

$$\alpha_{21}m_1\ddot{x}_1 + \alpha_{22}m_2\ddot{x}_2 + \alpha_{23}m_3\ddot{x}_3 + x_2 = 0$$

$$\alpha_{31}m_1\ddot{x}_1 + \alpha_{32}m_2\ddot{x}_2 + \alpha_{33}m_3\ddot{x}_3 + x_3 = 0$$

Substitute $\ddot{x}_1 = -\omega^2 x_1, \quad \ddot{x}_2 = -\omega^2 x_2$ and $\ddot{x}_3 = -\omega^2 x_3$

$$x_1 = \alpha_{11}m_1x_1 + \alpha_{12}m_2x_2 + \alpha_{13}m_3x_3$$

$$x_2 = \alpha_{21}m_1x_1 + \alpha_{22}m_2x_2 + \alpha_{23}m_3x_3$$

$$x_3 = \alpha_{31}m_1x_1 + \alpha_{32}m_2x_2 + \alpha_{33}m_3x_3$$

The above equation can be written in the matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} \alpha_{11}m_1 & \alpha_{12}m_2 & \alpha_{13}m_3 \\ \alpha_{21}m_1 & \alpha_{22}m_2 & \alpha_{23}m_3 \\ \alpha_{31}m_1 & \alpha_{32}m_2 & \alpha_{33}m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Substitute the value of influence coefficients in the above equation.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} \frac{4m}{3k} & \frac{2m}{3k} & \frac{m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{4m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{7m}{3k} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

To start the iteration process, assume the configuration in the first mode as

$x_1 = 1, x_2 = 2, x_3 = 3$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 11 \\ 32 \\ 41 \end{Bmatrix} = (11)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix}$$

Second iteration

$$\begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 13.5 \\ 42.16 \\ 53.32 \end{Bmatrix} = (13.5)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix}$$

Third iteration

$$\begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 14.19 \\ 44.76 \\ 56.81 \end{Bmatrix} = (14.19)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 14.3 \\ 45.16 \\ 57.13 \end{Bmatrix} = (14.3)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}$$

The ratio obtained is very close to the initial value.

$$\therefore \quad \begin{bmatrix} 1 \\ 3.15 \\ 3.99 \end{bmatrix} = 14.3\frac{m\omega^2}{3k} \begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix} \text{ or } \frac{14.3\, m\omega^2}{3k} = 1,\ \omega^2 = \omega_1^2 = \frac{3}{14.3}\frac{k}{m}$$

$$\therefore \quad \omega_1 = 0.458\sqrt{\frac{k}{m}} \text{ rad/s}$$

The first principal mode is given by $\begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}$

The second natural frequency and principal mode is found by using orthogonality principle.

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$

The value is converging to $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ and hence $\dfrac{3m\omega^2}{3k} = 1$

$\therefore \qquad \omega_2^2 = \dfrac{k}{m},\ \omega_2 = \sqrt{\dfrac{k}{m}}$ rad/s

and the second principle mode is $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$

To get the third mode, use orthogonality principle.

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

Substitute $A_1 = 1,\ A_2 = -1,\ B_1 = 3.158,\ B_2 = 0, C_1 = 4.0, C_2 = 1$

$$4m\ (-1)A_3 + 2m(0)B_3 + m(1)C_3 = 0,\ -4A_3 + C_3 = 0 \quad \therefore\ A_3 = \frac{+C_3}{4}$$

$4m(1)\ A_3 + 2m(1)\ B_3 + m(4)\ C_3 = 0,\ 4m\left(\dfrac{+C_3}{4}\right) + 2\ (3.158)\ B_3 + 4C_3 = 0,\ 6.316\ B_3 = -5C_3$

Then $\qquad \begin{matrix} A_3 \\ B_3 \\ C_3 \end{matrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \end{Bmatrix}$

When this is contained with the matrix equation for the second mode, it will yield the third mode.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$4m(1)A_2 + 2m(3.158)B_2 + m(3.99)C_2 = 0$$

$\therefore \qquad 4A_2 + 6.316\ B_2 + 3.99\ C_2 = 0$

$$A_2 = -1.58\ B_2 - C_2,\ B_2 = B_2 \text{ and } C_2 = C_2$$

The same can be written in matrix form as

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix} = \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix}$$

When this is combined with the matrix equation for first mode, it will converge to second mode.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

For first iteration, assume $x_1 = 1, x_2 = 1, x_3 = 1$.

$$\therefore \quad \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -7.32 \\ 1.68 \\ 4.68 \end{bmatrix} = \frac{7.32m\omega^2}{k}\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix}$$

For second iteration

$$\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.91 \\ 0.3864 \\ 2.31 \end{Bmatrix} = \frac{2.91m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.13 \\ 0.8 \end{Bmatrix}$$

For third iteration

$$\begin{Bmatrix} -1 \\ 0.13 \\ 0.8 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.13 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.48 \\ 0.218 \\ 2.14 \end{Bmatrix} = \frac{2.48m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix}$$

For fourth iteration

$$\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.92 \\ 0.1344 \\ 2.72 \end{Bmatrix} = \frac{2.92m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.04 \\ 0.93 \end{Bmatrix}$$

Assume the configuration in third mode as, $x_1 = 1, x_2 = 2, x_3 = 3$

Then first iteration

$$\begin{Bmatrix} -1 \\ 3.14 \\ 4.00 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix}\begin{Bmatrix} 1 \\ 3.14 \\ 4.00 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} 1.68 \\ -5.28 \\ 6.72 \end{Bmatrix} = \frac{1.68m\omega^2}{3k}\begin{Bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{Bmatrix}$$

Hence, the third principal mode $\begin{bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{bmatrix}$

The third natural frequency is given by $1.68\,\dfrac{m\omega^2}{3k} = 1,\ \omega^2 = \omega_3^2 = \dfrac{3}{1.68}\dfrac{k}{m} = 1.785\,\dfrac{k}{m}$

$$\therefore \qquad \omega_3 = 1.336\sqrt{\frac{k}{m}}\ \text{rad/s}$$

Hence the natural frequencies are

$$\omega_1 = 0.458\sqrt{\frac{k}{m}}\ \text{rad/s},\ \omega_2 = \sqrt{\frac{k}{m}}\ \text{rad/s},\ \omega_3 = 1.336\sqrt{\frac{k}{m}}\ \text{rad/s}$$

The principal modes are $\begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ and $\begin{Bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{Bmatrix}$