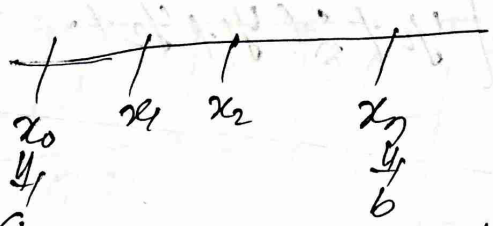
M:-2

Numerical Integration:

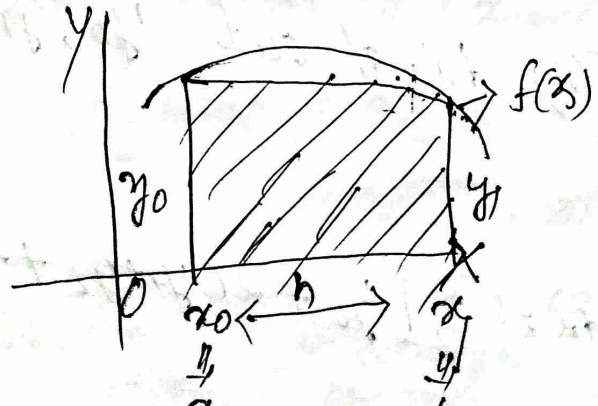
Introduction:

Numerical Integration is the process of computing the value of a definite integral from the tabulated values of the integrand.



Choosing interval size $h = \frac{b-a}{n}$. divide the interval $[a,b]$ in n intervals by means of $(n+1)$ equally spaced points $x_0 = a$, $x_i = x_0 + ih$, $i = 0, 1, 2 \cdots (n-1)$ $x_n = b$. Then finding out approximate value of $I = \int_{x_0}^{x_n} f(x)\,dx$, $f(x)$ = Integrand.

Trapezoidal Rule:- ($n=1$)



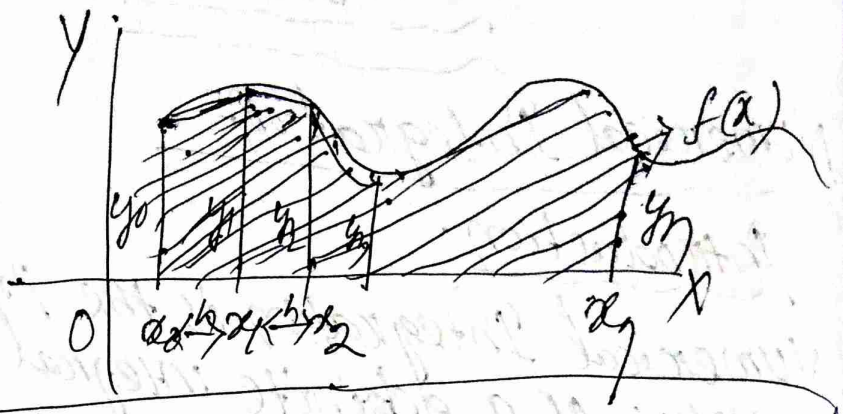
$h = \frac{b-a}{n}$

$h = \frac{b-a}{1} = b-a$

Take $n=1$, (two points, one interval)

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

For $n > 1$



$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

Step :-

① Take n = no. of Intervals

② calculate $h = \frac{b-a}{n}$

$x_n = b \rightarrow$ Upper limit

$x_0 = a \rightarrow$ Lower limit

③ find $x_0 = a$

$x_1 = a + h$

$x_2 = a + 2h$

⋮

$x_n = b = a + nh$

④ Using $y = f(x)$ find out $y_0, y_1, y_2, y_3 \cdots y_n$

⑤ $I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$

(Q) Evaluate $\int_1^4 (x^2+2x)dx$ by using Trapezoidal rule for $n=6$.

Solution:

Here given that $n=6$

$x_0=1$, $x_n=4$, $h=\frac{x_n-x_0}{n}=\frac{4-1}{6}=\frac{3}{6}=0.5$

x	1	1.5	2	2.5	3	3.5	4
y	3	5.25	8	11.25	15	19.25	24

$$\int_{x_0}^{x_n} f(x)dx = \int_1^4 (x^2+2x)dx$$

$$= \frac{0.5}{2}\left[3+2(5.25+8+11.25+15+19.25)+24\right]$$

$$= 36.125 \text{ (Ans)}$$

(Q) Calculate by Trapezoidal rule an approximate value of $\int_{-3}^{3} x^4 dx$ by taking seven equidistant points.

Solution: Here the range of integration $(-3,3)$ is subdivided into six equal parts by by seven equidistant points.

Here $x_0=-3=a$ $\quad h=\frac{b-a}{n}=\frac{3-(-3)}{6}=\frac{6}{6}=1$.

$x_n=3=b$

$n=6$

x	-3	-2	-1	0	1	2	3
$y=f(x)$	81	16	1	0	1	16	81

$$\int_a^b f(x)\,dx = \int_{-1}^{4} x^4\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4+y_5) + y_6\right]$$

$$\Rightarrow \frac{1}{2}\left[81 + 2(16+1+0+1+16) + 81\right]$$

$$\Rightarrow \frac{1}{2}\left[162 + 68\right]$$

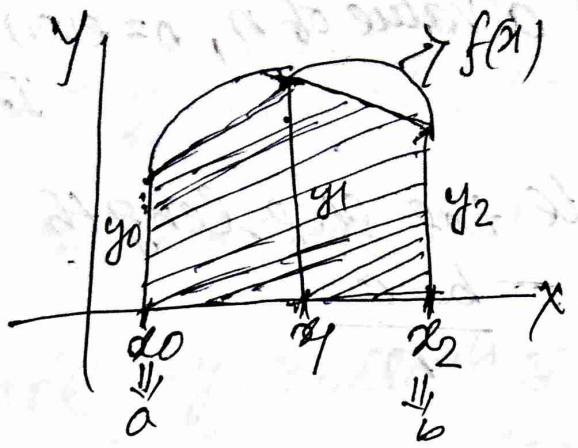
$$= 115 \text{ (Ans)}.$$

H.W
(Q) Calculate by Trapezoidal rule an approximate value of $\int_{a=0}^{b=4} \sqrt{64-x^3}\,dx$ by taking nine equidistant points.

H.W
(Q) Use Trapezoidal rule to evaluate $\int_1^2 \frac{dx}{x^2}$ corresponding to five Intervals.

H.W
(Q) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Trapezoidal rule for $n=6$.

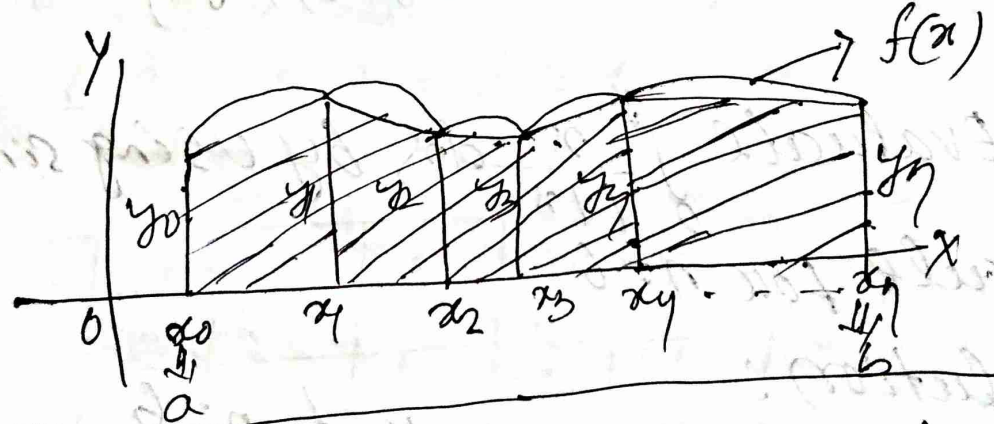
Simpson's $\frac{1}{3}$ Rule ($n=2$)



Take $n=2$ (Three Points, two Intervals).

$$I = \int_a^b f(x)\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

for $n > 2$ (when n is multiple of 2)
(2, 4, 6, 8 - - -)



$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots) + y_n\right]$$

step:

step:1: Take a value of n, n = even no. of Intervals.

step:2 calculate the step length

$$h = \frac{b-a}{n}$$

step:3 find $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$ ... $x_n = b = a+nh$.

step:4 using $y = f(x)$ find out $y_0, y_1, y_2 \cdots y_n$

step:5 $$\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots y_{n-2}) + y_n\right]$$

(Q) Evaluate $\int_0^1 \frac{x}{1+x}\,dx$ by using simpsons $\frac{1}{3}$rule for $n=6$

solution:

Here given that $n=6$

$a=0$, $b=1$, $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	0	1/7	2/8	3/9	4/10	5/11	6/12

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5) + 2(y_2+y_4) + y_6\right]$$

$$\int_0^1 \frac{x}{1+x}\,dx = \frac{1}{3\times 6}\left[0 + 4\left(\frac{1}{7}+\frac{3}{9}+\frac{5}{11}\right) + 2\left(\frac{2}{8}+\frac{4}{10}\right) + \frac{6}{12}\right]$$

$$= 0.6068 \text{ (Ans)}$$

(a) calculate by simpson's $\frac{1}{3}$ rule an approximate value of $\int_1^5 \frac{1}{x+2}\,dx$ by taking five equidistant points.

Solution: Here the range of Integration $(1, 5)$ is subdivided into four equal parts by five equidistant points.

Here $a=1$, $b=5$, $n=4$

$$h = \frac{b-a}{n} = \frac{5-1}{4} = \frac{4}{4} = 1$$

x	1	2	3	4	5
$y=\frac{1}{x+2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

$$\int_1^5 \frac{1}{x+2}\,dx = \frac{1}{3}\left[\frac{1}{3} + 4\left(\frac{1}{4}+\frac{1}{6}\right) + 2\times\left(\frac{1}{5}\right) + \frac{1}{7}\right]$$

$$= 0.8427 \text{ (Ans)}$$

H.W.

(Q) Evaluate $\int_0^6 \frac{1}{(1+x^2)}\,dx$ by using Simpson's $\frac{1}{3}$ rule for $n=6$.

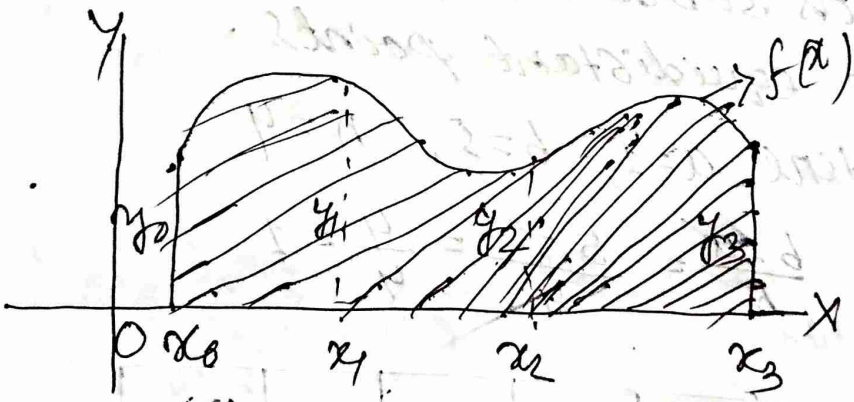
H.W

(Q) Calculate by Simpson's $\frac{1}{3}$ rule an approximate value of $\int_{4}^{5.2} \log x\,dx$ by taking seven equidistant points.

H.W

(Q) Evaluate $\int_0^1 e^{-x^2}\,dx$ by using Simpson's $\frac{1}{3}$ rule for $n=2$

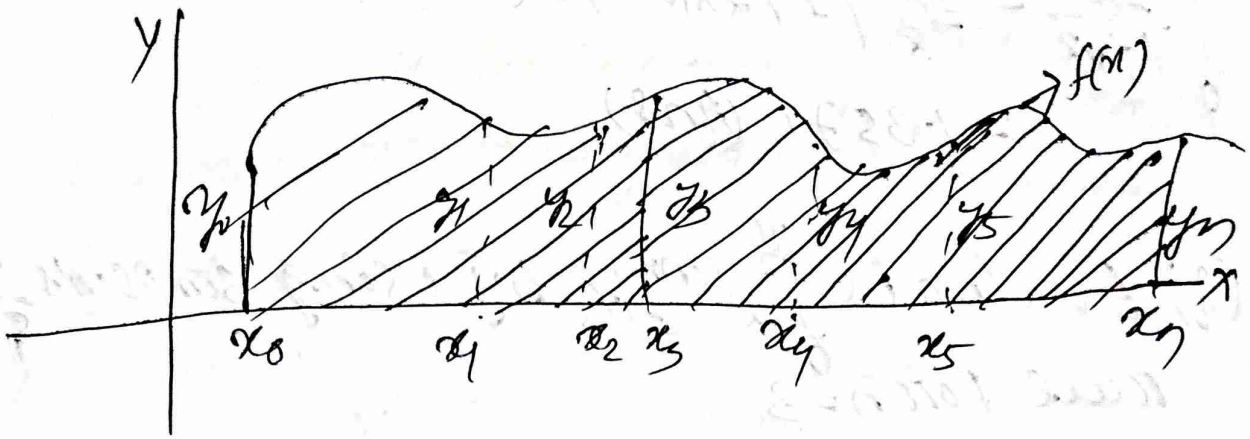
Simpson's $\frac{3}{8}$ rule :- ($n=3$)



Take $n=3$ (four points, three Intervals)

$$I = \int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

For $n > 3$ (when n is multiple of 3)
$(3, 6, 9 \cdots)$



$$I = \int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + \cdots) + y_n \right]$$

(Q) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$ rule for $n = 6$.

Solution: Here given that $n = 6$

$a = 0$, $b = 6$, $h = \frac{b-a}{n} = \frac{6-0}{6} = \frac{6}{6} = 1$

x	0	1	2	3	4	5	6
$y = f(x)$	1	0.5	0.2	0.1	0.0588	0.0285	0.027

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 2(y_3) + 3(y_1+y_2+y_4+y_5) + y_6\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3\times 1}{8}\left[1 + 2\times(0.1) + 3(0.5+0.2+0.0588+0.285) + 0.2\right]$$

$$\ldots\ldots = 1.3571 \text{ (Ans)}$$

(9) Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$ by using Simpson's $\frac{3}{8}$ rule for $n=3$

Solution:

Here given that $n=3$

$$a=0,\ b=\frac{\pi}{2},\ h=\frac{b-a}{n}=\frac{\frac{\pi}{2}-0}{3}=\frac{\pi}{6}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	1.64872	2.36320	2.71828

By Simpson's $\frac{3}{8}$ rule we have

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx = \frac{3}{8}\times\frac{\pi}{6}\left[1 + 3(1.64872+2.36320) + 2.7182\right.$$

$$= \frac{\pi}{16}\left[3.71828 + 0.3576\right] = 0.0911 \text{ (Ans)}$$

H.W

(Q) Evaluate $\int_0^3 \frac{1}{1+x}\,dx$ with $n=6$ by using Simpson's $\frac{3}{8}$ rule.

H.W

(Q) Calculate by Simpson's $\frac{3}{8}$ rule an approximate value of $\int_2^{5.2} \log x\,dx$ by taking seven equidistant points.

H.W

(Q) Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$ with $n=6$ by using Simpson's $\frac{3}{8}$ rule.

Gauss-Legendre quadrature Methods:

Let us take the definite integral given by

$$I(f) = \int_a^b f(x)\,dx \quad \text{——— (1)}$$

The Interval of Integration $[a,b]$ can be transformed to $[-1,1]$ by using the transformation

$$x = \frac{(b-a)}{2}t + \frac{b+a}{2}$$

substituting $x = \frac{(b-a)}{2}t + \frac{(b+a)}{2}$ in (1) we can obtain

$$I(f) = \int_a^b f(x)\,dx = \int_{-1}^{1} \phi(t)\,dt$$

Gauss-Legendre one-point formula ($n=0$):

$$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

Two point formula ($n=1$):

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Three point formula ($n=2$):

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

(Q) Evaluate approximately the Integral $I(f) = \int_0^1 \frac{1}{1+x}\,dx$ by 1-point, 2-point and 3-point Gauss-Legendre rules.

Solution:

First we transform the Interval $[0, 1]$ to the Interval $[-1, 1]$

$f(x) = \frac{1}{1+x}$, $a=0$, $b=1$

Let us put $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$$x = \frac{1-0}{2}t + \frac{1+0}{2} = \frac{t+1}{2}$$

$x=0$; $\frac{t+1}{2} = 0 \Rightarrow t+1=0 \Rightarrow t=-1$

$x=1$; $\frac{t+1}{2} = 1 \Rightarrow t+1=2 \Rightarrow t = 1-2 = -1$

$$x = \frac{t+1}{2}$$

Differentiating w.r.t "t" we get

$$\frac{dx}{dt} = \frac{1}{2} \Rightarrow dx = \frac{dt}{2}$$

$$f(x) = \frac{1}{1+x} = \frac{1}{1+\frac{t+1}{2}} = \frac{1}{\frac{t+3}{2}} = \frac{2}{t+3}$$

$$\int_0^1 \frac{1}{1+x}\,dx = \int_{-1}^{1} \frac{2}{t+3} \times \frac{dt}{2} = \int_{-1}^{1} \frac{dt}{t+3}$$

One-point formula:

$$\int_{-1}^{1} \frac{dt}{t+3} = 2f(0) = 2\left(\frac{1}{0+3}\right) = \frac{2}{3} = 0.666$$

Two-point formula:

$$\int_{-1}^{1} \frac{dt}{t+3} = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\left(\frac{-1}{\sqrt{3}}+3\right)} + \frac{1}{\left(\frac{1}{\sqrt{3}}+3\right)}$$

$$= \frac{\left(3+\frac{1}{\sqrt{3}}+3-\frac{1}{\sqrt{3}}\right)}{\left(3-\frac{1}{\sqrt{3}}\right)\left(3+\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{6}{\left(9-\frac{1}{3}\right)} = \frac{18}{26} = 0.6923$$

Three-point formula:

$$\int_{-1}^{1} \frac{dt}{t+3} = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\frac{1}{(-\sqrt{\frac{3}{5}}+3)} + \frac{8}{9}\times\frac{1}{3} + \frac{5}{9}\frac{1}{(\sqrt{\frac{3}{5}}+3)}$$

$$= \frac{5}{9}\left\{\frac{1}{(3-\sqrt{\frac{3}{5}})} + \frac{1}{(3+\sqrt{\frac{3}{5}})}\right\} + \frac{8}{27}$$

$$= \frac{5}{9}\times\frac{6}{(9-\frac{3}{5})} + \frac{8}{27}$$

$$= \frac{5}{9}\times\frac{30}{42} + \frac{8}{27}$$

$$= \frac{25}{63} + \frac{8}{27} = \frac{75+56}{189} = \frac{131}{189} = 0.6931$$

(Q) Evaluate approximately the Integral $I(f) = \int_{-1}^{1} e^{-x^2}\,dx$ by 1-point, 2-point, and 3-point Gauss-Legendre rules.

Solution:

Given that function satisfies Gauss-Legendre rule.

$$f(x) = e^{-x^2}$$

one-point formula:

$$\int_{-1}^{1} e^{-x^2}\,dx = 2f(0) = 2\cdot e^{-0^2} = 2\cdot 1 = 2.$$

Two-point formula:

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} e^{-x^2}\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= e^{-1/3} + e^{-1/3} = 2\cdot e^{-1/3} = 2\times(0.7165)$$

$$= 1.433.$$

Three-point formula:

$$\int_{-1}^{1} e^{-x^2}\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9} e^{-.315} + \frac{8}{9}\times 1 + \frac{5}{9} e^{-.315}$$

$$= \frac{10}{9} e^{-.315} + \frac{8}{9}$$

$$= 1.4986 \text{ (Ans)}.$$

H.W (Q) Evaluate approximately the Integral $I(f) = \int_2^4 (x+1)\,dx$ by 1-point, 2-point and 3-point Gaces-Legendre rules.

H.W (Q) Evaluate $\int_0^{\pi/2} \sin x\,dx$ by the two-point Gauss-Legendre rules.

H.W (Q) Evaluate $\int_0^1 e^{x^2}\,dx$ by the two-point Gauss-Legendre rules.

Numerical solution of IVP's of ODE's :

Euler's method :

Let us take the Initial value problem given by

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \quad \text{---(1)}$$

suppose that we have to solve the equation (1) for values of y at $x = x_i$

Where $x_i = x_0 + ih$ for $i = 1, 2, 3 \ldots$

Now, $\frac{dy}{dx} = f(x,y)$

$\Rightarrow dy = f(x,y)\,dx$

Integrating on both side we get

$$\Rightarrow \int_{y_0}^{y_1} dy \simeq \int_{x_0}^{x_1} f(x,y)\,dx$$

Assume that $f(x,y) \simeq f(x_0, y_0)$ in $[x_0, x_1]$ this gives

$$\int_{y_0}^{y_1} dy \simeq \int_{x_0}^{x_1} f(x_0, y_0)\,dx$$

$$\Rightarrow [y]_{y_0}^{y_1} \simeq f(x_0, y_0)\,[x]_{x_0}^{x_1}$$

$$\Rightarrow y_1 - y_0 \simeq (x_1 - x_0) f(x_0, y_0)$$

$$\Rightarrow y_1 \simeq y_0 + h f(x_0, y_0)$$

Proceeding in this way, we can get

$$y_2 \approx y_2 + hf(x_2, y_2)$$

$$\vdots$$

In general

$$\boxed{y_{k+1} = y_k + hf(x_k, y_k) \text{ for } k = 0, 1, 2 \cdots}$$

This is known as Euler's method for approximate solution of the initial value problem.

(Q) Using Euler's method find out $y(0.2)$ given that $\frac{dy}{dx} = x+y$; $y(0) = 1$, $h = 0.1$.

Solution: $\frac{dy}{dx} = x+y$; $y(0) = 1$

Here $f(x, y) = x+y$

$x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$f(x_0, y_0) = x_0 + y_0 = 0+1 = 1$

$y_1 = y(0.1) = y_0 + hf(x_0, y_0)$

$= 1 + 0.1(1) = 1.1$.

$x_1 = 0.1$, $y_1 = 1.1$

$f(x_1, y_1) = 0.1 + 1.1 = 1.2$

$y_2 = y(0.2) = y_1 + hf(x_1, y_1)$

$= 1.1 + 0.1 \times (1.2) = 1.22$.

(a) Determine y for $x = 0.1, 0.2, 0.3, 0.4, 0.5$ where y is the solution of the differential equation $\frac{dy}{dx} = 2(y+1)$; $y(0) = 0$ by using Euler's method with $h = 0.1$. Find the exact solution and compare your numerical result with the exact solution.

Solution :

Here $f(x, y) = 2(y+1)$

$x_0 = 0$, $y_0 = 0$

$h = 0.1$

$x_1 = 0.1$

$f(x_0, y_0) = 2(y_0 + 1) = 2(0+1) = 2$

$y(0.1) = y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.1 \times 2 = 0.2$

$x_2 = 0.2$

$f(x_1, y_1) = 2(y_1 + 1) = 2(0.2 + 1) = 2.4$

$y(0.2) = y_2 = y_1 + h f(x_1, y_1)$

$= 0.2 + (0.1) \times (2.4) = 0.44$

$x_3 = 0.3$

$f(x_2, y_2) = 2(y_2 + 1) = 2(0.44 + 1) = 2.88$

$y(0.3) = y_3 = y_2 + h f(x_2, y_2)$

$= 0.44 + (0.1) \times (2.88) = 0.728$

$x_4 = 0.4$

$f(x_3, y_3) = 2(y_3 + 1) = 2(0.728 + 1) = 3.456$

$$y(0.4) = y_4 = y_3 + hf(x_3, y_3)$$
$$= 0.728 + (0.1) \times (3.456)$$
$$= 1.0736.$$

$x_5 = 0.5$

$f(x_4, y_4) = 2(y_4 + 1) = 2(1.0736 + 1)$
$= 4.1472$

$$y(0.5) = y_5 = y_4 + hf(x_4, y_4)$$
$$= 1.0736 + (0.1)(4.1472)$$
$$= 1.48832$$

$\frac{dy}{dx} = 2(y+1)$

Using variable separable method we get

$dy = 2(y+1)\,dx$

$\Rightarrow \frac{dy}{y+1} = 2\,dx$

Integrating on both side we get

$\Rightarrow \int \frac{dy}{y+1} = \int 2\,dx$

$\Rightarrow \ln(y+1) = 2x + C$ ——— (1)

putting $x = 0$, $y = 0$ in eqn (1) we get

$\Rightarrow \ln(0+1) = 2 \times 0 + C$

$\Rightarrow C = 0$

putting $C = 0$ in eqn (1) we get

$\ln(y+1) = 2x$

$\Rightarrow y+1 = e^{2x}$

$\Rightarrow y = e^{2x} - 1$

$y(0.1) = e^{0.2} - 1 = 0.2214027$

$y(0.2) = e^{0.4} - 1 = 0.4918246$

$y(0.3) = e^{0.6} - 1 = 0.822118$

$y(0.4) = e^{0.8} - 1 = 1.2255409$

$y(0.5) = e - 1 = 1.7182818$

H.W

(Q) Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(0.02)$ by Euler's method taking $h = 0.01$.

H.W

(Q) Use Euler's method with step size 0.3 to compute the approximate y-value $y(0.9)$ of the solution of the initial value problem $y' = x^2$, $y(0) = 1$. Find the exact solution and compare your numerical result with the exact solution.

H.W

(Q) Given $\frac{dy}{dx} = e^x$, $y(0) = 1$. Compute $y(3)$ by Euler's method taking $h = 1$.

H.W

(Q) solve $\frac{dy}{dx} = xy$, $y(0) = 0$ by Euler's method taking $h = 0.2$ and find out $y(0.4)$.

Modified Euler's method :-

Given $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Modified Euler's formula is given by

$$y_{n+1}^{(n+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(n)})\right]$$

where $n = 0, 1, 2 \ldots$

$n = 0, 1, 2 \ldots$

where $y_{n+1}^{(0)} = y_n + h f(x_n, y_n)$ → using Euler's method

$n = 0, 1, 2 \ldots$

(a) By taking step length $(h) = 0.05$ find $y(0.05)$ by modified Euler's method for IVP $\frac{dy}{dx} = x + y$, $y(0) = 1$ and by taking correct up to 3-decimal places.

Solution :-

Given that $\frac{dy}{dx} = x + y$

$f(x, y) = x + y$

$x_0 = 0, y_0 = 1, \quad h = 0.05$

put $x = 0$, $n = 0$ then find

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1 + 0.05[0 + 1]$

$= 1.05$

By using modified Euler's method is

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 1.05)]$$

$$= 1 + \frac{0.05}{2}[(0+1) + (0.05+1.05)]$$

$$= 1.0525.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 1.0525)]$$

$$= 1 + \frac{0.05}{2}[(0+1) + (0.05+1.0525)]$$

$$= 1.0525$$

So these two approximate values are same and correct up to 3-decimal places.

$$y(0.05) = 1.052 \quad \text{(Ans)}$$

H.W

(Q) Using modified Euler's method find the value of $y(0.2)$ up to 4 places of decimals.

$$\frac{dy}{dx} = -xy^2; \quad y(0) = 2, \text{ taking } h = 0.1$$

H.W

(Q) By taking step length 0.1 find $y(0.1)$ by modified Euler's method for IVP

$$\frac{dy}{dx} = x + 2y; \quad y(0) = 0; \quad h = 0.1.$$

H.W

(Q) Using Modified Euler's method find the value of $y(1.2)$ correct to three decimal places

$$\frac{dy}{dx} = \log(x+y), \quad y(1) = 2, \quad h = 0.2$$

Runge-Kutta Method: (R-K Method)

(1) First order R-K Method:

Consider the first order equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

The Euler's formula for the first approximation to the solution of the above differential equation is given by

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Euler's Method is the Runge-Kutta method of 1st order.

(2) Second order R-K Method:

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

Where $K_1 = hf(x_n, y_n)$

$$K_2 = hf(x_n + h, y_n + K_1)$$

(3) Third order R-K Method:

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

Where $K_1 = hf(x_n, y_n)$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = hf(x_n + h, y_n + K_2)$$

(4) fourth-order R-K method:-

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_n + h, y_n + K_3)$$

(Q) By taking step length of 0.1 find $y(0.2)$ By R-K method of order 2. Initial value Problem $\frac{dy}{dx} = x + y$, $y(0) = 1$

Solution:-

Given that $f(x, y) = x + y$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$K_1 = hf(x_0, y_0) = 0.1[0 + 1] = 0.1$$

$$K_2 = hf(x_0 + h, y_0 + K_1)$$

$$= 0.1[0.1 + 1.1] = 0.12$$

$$y(0.1) = y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 1 + \frac{1}{2}[0.1 + 0.12] = 1.11$$

Now evaluate $y(0.2)$ =

$K_1 = hf(x_1, y_1) = 0.1[0.1 + 1.11]$

$= 0.1[1.21] = 0.121$

$K_2 = hf(x_1 + h, y_1 + K_1)$

$= 0.1[0.2 + 1.11 + 0.121]$

$= 0.1431$

So by R-K method

$y(0.2) = y_2 = y_1 + \frac{1}{2}(K_1 + K_2)$

$= 1.11 + \frac{1}{2}[0.121 + 0.1431]$

$= 1.24205$

(@) Using Runge-Kutta method of fourth order determine $y(0.1)$ and $y(0.2)$ correct to four decimal places given that $\frac{dy}{dx} = y - x$ where $y(0) = 2$; $h = 0.1$.

Solution: $\frac{dy}{dx} = f(x, y) = y - x$

$x_0 = 0$, $y_0 = 2$, $h = 0.1$ To obtain $y(0.1)$

we have $K_1 = hf(x_0, y_0) = 0.1(y_0 - x_0)$

$= 0.1(2 - 0) = 0.2$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= 0.1\left[y_0 + \frac{K_1}{2} - (x_0 + \frac{h}{2})\right]$

$= 0.1[2.0 + 0.1 - 0.05]$

$= 0.205$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= 0.1\left[y_0 + \frac{K_2}{2} - (x_0 + \frac{h}{2})\right]$

$= 0.1[2.1025 - 0.05] = 0.20525$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1[y_0 + K_3 - (x_0 + h)]$

$= 0.1[2.205 - 0.1] = 0.2105$

$y(0.1) = y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 2 + \frac{1}{6}(0.2 + 2 \times 0.205 + 2 \times 0.20525 + 0.2105)$

$= 2.205$

Next to obtain $y(0.2)$ take $x_1 = 0.1$

$y_1 = 2.205$ hence

$K_1 = hf(x_1, y_1) = 0.1[2.205 - 0.1] = 0.2105$

$K_2 = hf[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}]$

$= 0.1[2.205 + 0.10525 - (0.1 + 0.05)]$

$= 0.2160$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.1[2.205 + 0.108 - (0.1 + 0.05)]$$

$$= 0.2163$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$= 0.1[2.205 + 0.2163 - (0.1 + 0.1)] = 0.22213$$

$$y(0.2) = y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 2.205 + \frac{1}{6}[0.2105 + 2 \times 0.2160 + 2 \times 0.2163 + 0.22213]$$

$$= 2.205 + 0.2162 = 2.4212 \text{ (Ans)}$$

H.W.

(Q) Solve the equation $\frac{dy}{dx} = -y$, $y(0) = 1$, $h = 0.1$ for values of y at $x = 0.1$ and $x = 0.2$ using Runge-Kutta Method of (i) order two (ii) order three (iii) order four.

H.W

(Q) Using Runge-Kutta method of fourth order determine $y(0.6)$. Initial value problem

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0, \quad h = 0.2$$

H.W

(Q) Apply the second order and fourth order Runge-Kutta method to solve

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1.$$ Taking step size $h = 0.1$ and determine $y(0.1)$ and $y(0.2)$

Multistep Method:-

A multistep method uses, in each step, values from two or more previous steps. These methods are motivated by the expectation that the additional information will increase accuracy and stability.

Adam's-Bashforth Method:-(ABM)

Given $\frac{dy}{dx} = f(x,y)$ and $y(x_0) = y_0$

→ Adam's-Bashforth predictor formula is

$$y_{n+1,P} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$
$$= y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

→ Adam's-Bashforth corrector formula is

$$y_{n+1,C} = y_n + \frac{h}{24}[9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2}]$$
$$= y_n + \frac{h}{24}[9y'_{n+1} - 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

(Q) Using Adam's-Bashforth method find $y(1.4)$ given $\frac{dy}{dx} = f(x,y) = x^2(1+y)$

$y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$

Solution:-

Given $y' = x^2(1+y)$; $(x_0, y_0) = (1, 1)$

$(x_1, y_1) = (1.1, 1.233)$, $(x_2, y_2) = (1.2, 1.548)$

$(x_3, y_3) = (1.3, 1.979)$ and $h = 0.1$.

As per Adam's - Bashforth predictor formula

$$y_{4,P} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

Where $y_0' = x_0^2(1+y_0) = (1)^2(1+1) = 2$.

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$.

$y_2' = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.66912$

$y_3' = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.03451$

$$y_{4,P} = 1.979 + \frac{0.1}{24}\{55(5.03451) - 59(3.66912) + 37(2.70193) - 9(2)\}$$

$= 2.5722974$

Hence $y_4' = x_4^2(1+y_{4,P})$

$= (1.4)^2\{1+2.5722974\} = 7.0017029$

Next, the corrector formula is

$$y_{4,C} = y_3 + \frac{h}{24}\{9y_4' + 19y_3' - 5y_2' + y_1'\}$$

$$= 1.979 + \frac{0.1}{24}\{9(7.0017029) + 19(5.03451) - 5(3.66912) + 2.70193\}$$

$= 2.5749473$.

Thus $y(1.4) = 2.575$ (Ans).