

## Probability:-

Probability is a concept which numerically measures the degree of uncertainty and there of certainty of the occurrence of events.

If an event A can happen in  $m$  ways, and fail in  $n$  ways where all ways are being equally likely to occur, then the probability of the happening of event A is defined as

$$P(A) = \frac{\text{Number of favourable cases}}{\text{Total number of mutually exclusive and equally likely cases}}$$

$$= \frac{m}{m+n} = p \text{ (say)}$$

While that of its failing is defined as  $P(\text{not } A)$

$$\text{or } P(A^c) = \frac{n}{m+n} = q \text{ (say)}$$

$$\text{Thus } P(A) + P(A^c) = p + q = \frac{m}{m+n} + \frac{n}{m+n} = 1$$

$$p + q = 1$$

From above it may be noted that  $P(A) = p$  is such that  $0 \leq p \leq 1$ .  $P(A^c) = q$  is called complementary event. Also  $0 \leq q \leq 1$ .

For instance on tossing a coin the probability of getting head is  $\frac{1}{2}$ .

## Exhaustive events or sample space (2) :-

The set of all possible outcomes of a single performance of an experiment is called exhaustive events or sample space and its elements are called the sample points.

Ex: (i) If we toss a coin twice

$$\text{then } S = \{HH, HT, TH, TT\}$$

(ii) If we roll a die once we have the corresponding sample space  $S = \{1, 2, 3, 4, 5, 6\}$ .

(iii) If we toss a coin once then sample space  $S = \{H, T\}$ . Two outcomes head and tail constitute one exhaustive event because no other outcome is possible.

## Mutually exclusive events :-

Two events are known as mutually exclusive when the occurrence of one of them excludes the occurrence of the other.

e.g. while tossing a coin we either get head or tail but not both.

## Favourable events :-

The events, which ensure the required happening, are said to be favourable events.

Ex: In throwing a die so have the even

numbers 2, 4, and 6 are favourable cases.

Definition:-  
 Let  $\Omega$  be a finite sample space and  $E$  be an event. Then the probability of the event  $E$  is given by  $P(E) = \frac{|E|}{|\Omega|}$  where  $|E|$  and  $|\Omega|$  denote the cardinality of  $E$  and  $\Omega$ .

### Algebra of events:-

The algebra of events corresponding to algebra of sets, using set operations on events in  $\Omega$ . Thus if  $A$  &  $B$  are two events, then

- (1)  $A \cup B$  is the event either  $A$  or  $B$  or both.
- (2)  $A \cap B$  or  $A \bar{B}$  is the event both  $A$  and  $B$ .
- (3)  $\Omega \setminus A$  or  $A^*$  is the event not  $A$ .
- (4)  $A \bar{B}$  is the event  $A$  but not  $B$ .
- (5) If the sets  $A$  and  $B$  are disjoint e.g.  $A \bar{B} = \emptyset$

(Q) Find the probability of getting 3 on rolling a die.

Sol :- Sample space ( $\Omega$ ) = {1, 2, 3, 4, 5, 6, 8}

Number of favourable event = 1

i.e. {3}

Total number of outcomes = 6

Thus Probability  $P = \frac{\text{no. of favourable event}}{\text{no. of outcomes}} = \frac{1}{6}$

(Q) Find the probability of throwing 9 with two dice.

Sol): Total number of possible ways of throwing two dice  $= 6 \times 6 = 36$   
Number of way getting 9 i.e.  
 $(3+6), (4+5), (5+4), (6+3) = 4$   
Thus, Probability  $P = \frac{4}{36} = \frac{1}{9}$

(Q) Consider the experiment of rolling a die. Let A be event getting a prime number. B be the event getting an odd number. Write the set representing the events (i) A or B (ii) A and B (iii) A but not B (iv) not A.

Sol): Here  $S = \{1, 2, 3, 4, 5, 6\}$   
 $A = \{2, 3, 5\}, B = \{1, 3, 5\}$

$$(i) A \text{ or } B = A \cup B = \{1, 2, 3, 5\}$$

$$(ii) A \text{ and } B = A \cap B = \{3, 5\}$$

$$(iii) A \text{ but not } B = A \setminus B = \{2\}$$

$$(iv) \text{ not } A = A' = \{1, 4, 6\}.$$

(Q) A coin is tossed three times, consider the following events.

A: No head appears; B: Exactly one head appears and C: At least two heads appear  
Do they form a set of mutually exclusive events?

Sol): The sample space of the experiment is  $S = \{HHH, HHT, HTT, THT, TTH, TTT\}$

$$\text{and } A = \{TTT\}, B = \{HHT, THT, TTH\}$$

$$C = \{HHT, HTT, THH, HHH\}$$

$$A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset$$

Therefore the events are pair-wise disjoint i.e. they are mutually exclusive events.

Note:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} ; P(B) \neq 0$$

$$\Rightarrow P(B|A) = \frac{P(B \cap A)}{P(A)} ; P(A) \neq 0$$

$$\Rightarrow P(A) + P(A^c) = 1$$

(Q) A and B are two events such that

$$(Q) A \text{ and } B \text{ are two events such that } P(A) = 0.3, P(B) = 0.4 \text{ and } P(A \cup B) = 0.6$$

$$\text{find (i) } P(A|B) \text{ (ii) } P(B|A^c)$$

Solution -

$$(i) P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{--- (1)}$$

$$\text{Now } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow 0.6 = 0.3 + 0.4 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 0.7 - 0.6 = 0.1$$

$$\text{From (1)} P(A|B) = \frac{0.1}{0.4} = \frac{1}{4}$$

$$(ii) P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)}$$

$$= \frac{P(B) - P(A \cap B)}{1 - P(A)}$$

$$= \frac{0.4 - 0.1}{1 - 0.3}$$

$$= \frac{0.3}{0.7} = \frac{3}{7}$$

Conditional Probability:

Let  $A$  and  $B$  be two events associated with the same sample space of a random experiment. Then the probability of occurrence of  $A$  under the condition that  $B$  has already occurred at  $P(B) \neq 0$  is called conditional probability.

It is denoted by  $P(A|B)$  and defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)} \text{ where } P(B) \neq 0$$

$$\text{and } P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)} \text{ where } P(A) \neq 0$$

Property :-

$$\rightarrow 0 \leq P(A|B) \leq 1. \text{ since } A \cap B \subseteq B; \text{ therefore}$$

$$P(A \cap B) \leq P(B)$$

$$\rightarrow P(A|B) = 0 \text{ iff } A \cap B = \emptyset$$

$$\rightarrow \text{In particular } P(\emptyset|B) = 0$$

$$\rightarrow P(A|B) = 1 \text{ iff } P(A \cap B) = P(B)$$

Let  $A$  and  $B$  be events of a sample space  $S$

of an experiment, then we have

$$P(C|B) = P(B|B) = 1$$

If  $A$  and  $B$  are any two events of  $S$  and  $E$  is another event of  $S$ . Such that  $P(E) \neq 0$

$$\text{then } P[A \cup B|E] = P(A|E) + P(B|E) - P(A \cap B|E)$$

if  $A$  and  $B$  are disjoint events then

$$P[A \cup B|E] = P(A|E) + P(B|E)$$

$$\rightarrow P(AC|B) = 1 - P(A|B)$$

$$\rightarrow P(A|B) + P(AC|B) = 1$$

$\rightarrow$  If  $A_1 \subseteq A_2$  then  $P(A_1|B) \leq P(A_2|B)$  where  $A_1$  and  $A_2$  are events of a sample space.

$\rightarrow$  If  $P(B) \neq 1$ , then  $P(A|B) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

$\rightarrow$  If  $A$  and  $B$  are mutually exclusive events and  $P(A \cup B) \neq 0$  then

$$P(A|(A \cup B)) = \frac{P(A)}{P(A) + P(B)}$$

(Q) A pair of dice is rolled, find  $P(A|B)$  if

A: 2 appears on at least one die

B: sum of numbers appearing on die is 6.

Solution:

We have  $A = \{(2,1), (2,2), (2,3), (2,4), (2,6), (1,2), (3,2), (4,2), (5,2), (6,2)\}$

$$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

$$A \cap B = \{(2,4), (4,2)\}$$

$$P(A \cap B) = \frac{2}{36}$$

$$P(B) = \frac{5}{36}$$

$$\text{Therefore } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

(Q) If A and B are two events  $P(A) = 0.6$ ,  $P(B) = 0.5$  and  $P(AnB) = 0.2$  then find  $P(A|B^c)$

Solution :

$$P(A|B^c) = \frac{P(A) - P(AnB)}{P(B^c)}$$

$$= \frac{0.6 - 0.2}{0.5} = \frac{4}{5}$$

(Q) Let A and B be two events such that  $P(A) = 0.6$ ,  $P(B) = 0.2$  and  $P(A|B) = 0.5$ . Then find  $P(A|B^c)$

Soln : By definition of conditional Probability

$$P(A|B) = \frac{P(AnB)}{P(B)} \Rightarrow P(AnB) = P(A|B) \cdot P(B)$$

$$\Rightarrow P(AnB) = 0.5 \times 0.2 = 0.1$$

$$\text{Hence } P(A|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)}$$

$$= \frac{P((A \cup B)^c)}{P(B^c)}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)} \quad \begin{aligned} \Rightarrow P(A \cup B) &= P(A) + P(B) - P(AnB) \\ &\Rightarrow P(A \cup B) = 0.6 + 0.2 - 0.1 = 0.7 \end{aligned}$$

$$= \frac{1 - 0.7}{1 - 0.2} = \frac{0.3}{0.8} = \frac{3}{8}$$

(Q) If  $P(A) = \frac{1}{3}$ ,  $P(B) = \frac{1}{4}$ ,  $P(A \cup B) = \frac{1}{2}$   
determine (i)  $P(B|A)$  (ii)  $P(A|B^c)$

Solution:

Given that  $P(A) = \frac{1}{3}$ ,  $P(B) = \frac{1}{4}$ ,  $P(A \cup B) = \frac{1}{2}$

We know that  $P(B) + P(B^c) = 1$ .

$$\Rightarrow P(B^c) = 1 - P(B) \\ = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{4+3-6}{12} = \frac{1}{12}$$

$$(i) P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

$$(ii) P(A|B^c) = \frac{P(A) - P(A \cap B)}{P(B^c)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{\frac{1}{3} - \frac{1}{12}}{\frac{3}{4}}$$

$$= \frac{\frac{3}{12} - \frac{1}{12}}{\frac{3}{4}} = \frac{\frac{2}{12}}{\frac{3}{4}} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{1}{9}$$

H.W  
 Q) If  $P(A) = \frac{3}{5}$ ,  $P(B) = \frac{2}{5}$  and  $P(AB) = \frac{1}{5}$   
 find (i)  $P(A \cup B)$  (ii)  $P(A|B)$  (iii)  $P(B|A)$   
 (iv)  $P(A|B^c)$

Independent Events:

Two events A and B are Independent if  
 $P(AB) = P(A) \cdot P(B)$ .

Q) If A and B are two Independent events with  $P(A) = \frac{1}{3}$  and  $P(B) = \frac{3}{4}$  then find  $P(AB)$ .

Solution:

Since A and B are independent events then  
 since  $P(AB) = P(A) \cdot P(B)$   
 $= \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

Q) If  $P(A) = 0.6$ ,  $P(B|A) = 0.5$  find  $P(A \cup B)$ . If  
 A and B are Independent.

Solution:

$$P(B|A) = \frac{P(AB)}{P(A)}$$

$$\Rightarrow P(AB) = P(A) \cdot P(A|B) = 0.6 \times 0.5 = 0.30$$

$$\text{Also } P(AB) = P(A) \cdot P(B).$$

$$\Rightarrow 0.30 = 0.6 \times P(B)$$

$$\Rightarrow P(B) = \frac{0.30}{0.6} = 0.5$$

$$\text{Now } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.5 - 0.30$$

$$= 1.1 - 0.30 = 0.8 = \frac{8}{10} = \frac{4}{5}$$

Baye's Theorem:

Let the sample space  $S$  be divided into  $n$  disjoint sets  $E_1, E_2, \dots, E_n$  whose union is  $S$ . These  $n$  sets are the events of an experiment.

Statement:

If  $E_1, E_2, \dots, E_n$  be  $n$  non-empty events which constitute a partition of sample space  $S$  and  $A$  is an arbitrary event with  $P(A) > 0$ ,

$$P(E_i | A) = \frac{P(E_i) \cdot P(A|E_i)}{\sum_{i=1}^n P(E_i) \cdot P(A|E_i)} ; i=1, 2, 3, \dots, n$$

Proof:

Here  $E_i \cap E_j = \emptyset$ ; if  $i \neq j$ ,  $i, j = 1, 2, \dots, n$   
 i.e. the events  $E_1, E_2, \dots, E_n$  are mutually disjoint  
 and  $S = E_1 \cup E_2 \cup \dots \cup E_n$

We know that for any event A.

$$\begin{aligned}A &= A \cap S = A \cap (E_1 \cup E_2 \cup \dots \cup E_n) \\&= (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)\end{aligned}$$

Since  $E_i \cap E_j = \emptyset$ ;  $i \neq j$  therefore

$$(A \cap E_i) \cap (A \cap E_j) = \emptyset$$

where  $A \cap E_i \subset E_i$  and  $A \cap E_i \subset E_j$ ,  $i, j = 1, 2, \dots, n$

$$\begin{aligned}P(A) &= P[(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)] \\&= P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n) \\&= \sum_{i=1}^n P(A \cap E_i) = \sum_{i=1}^n P(E_i) \cdot P(A|E_i)\end{aligned}$$

By formula of conditional probability

We have

$$\begin{aligned}P(E_i|A) &= \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i) \cdot P(A|E_i)}{P(A)} \\&= \frac{P(E_i) \cdot P(A|E_i)}{\sum_{i=1}^n P(E_i) \cdot P(A|E_i)}, \quad i = 1, 2, 3, \dots, n\end{aligned}$$

- (Q) Bag I contains 3 red and 4 black balls & Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag-II.

Solution:-

Let  $E_1$  be the event of choosing the Bag-I.

Let  $E_2$  be the event of choosing the Bag-II.

Let  $A$  be the event of drawing a red ball.  
We have to find  $P(E_2|A)$  i.e. probability of  
drawing a ball from Bag-II, given that  
it is red.

Since, probability of choosing each bag  
is same, we have  $P(E_1) = P(E_2) = \frac{1}{2}$   
Also  $P(A|E_1) = P(\text{drawing a red ball from bag I})$   
 $= \frac{3}{7}$

and  $P(A|E_2) = P(\text{drawing a red ball from bag II})$   
 $= \frac{5}{11}$

Using Baye's theorem,  $P(E_2|A) =$

$$\frac{P(E_2) \cdot P(A|E_2)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{35}{68}$$

## Random variable : (R.V.)

If  $S$  be a sample space then function  $X: S \rightarrow R$  is called a random variable.

There are two types of random

(i) Discrete random variable

(ii) Continuous random variable.

### (i) Discrete random variable:

A random variable  $X$ , which can take only a finite number of values in an interval of the domain is called discrete random variable.

Ex: (1) Number appearing on top of a dice when it is thrown.

(2) The number of telephone calls received per day.

(3) Number of mistakes in a page.

(4) The number of defective in a sample of electric bulbs.

### (ii) Continuous random variable:

If a random variable can assume all real values at a given interval. It is called a continuous random variable.

Ex: Height, weight, temperature, time are continuous random variable.

## Discrete Probability Distribution:

If a random variable  $X$  can assume a discrete set of values say  $x_1, x_2, x_3, \dots, x_n$  with respect probabilities  $p_1, p_2, p_3, \dots, p_n$  such that.

$$p_1 + p_2 + p_3 + \dots + p_n = 1 \text{ i.e. } \sum p_i = 1 \text{ then}$$

occurrences of values  $x_i$  with respective probabilities  $p_i$  is called the discrete probability distribution of  $X$ .

Ex:- In a throw of a pair of dice the sum ( $X$ ) is the discrete random variable which is an integer between 2 and 12 with probability  $P(X)$  given as

$X$	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

This constitutes a discrete probability distribution.

Probability function or probability mass function (P.m.f):

probability function or probability mass function (P.m.f) of a random variable  $X$  is mathematical function  $P(x)$  which gives the probabilities corresponding to different possible discrete set of values say  $x_1, x_2, x_3, \dots, x_n$  of variable  $X$  i.e.  $P(x_i) = P(X=x_i)$  = probability that random variable  $X$  assumes value  $x_i$ .

The function  $P(x)$  satisfies the condition

(i)  $P(x_i) \geq 0$

(ii)  $\sum P(x_i) = 1$

(Q) Find the probability distribution of  $X$ .  
the number of tails in two tosses  
of a coin.

Solution:-

In two tosses of a coin, the number of tails is 0, 1, 2.

$$\text{Now } P(X=0) = P(\text{no tail}) = P\{\text{hh}\} = \frac{1}{4} \geq 0$$

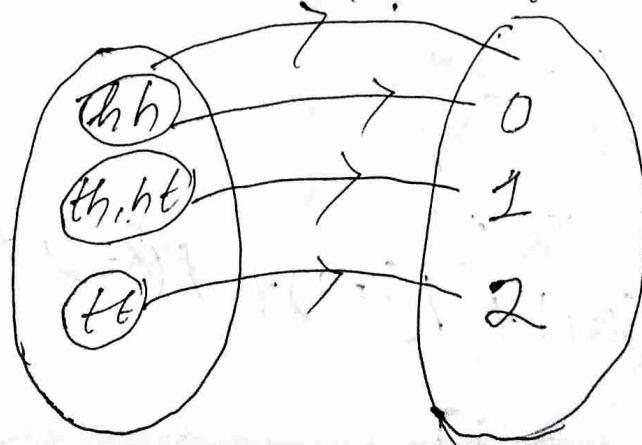
$$\begin{aligned} P(X=1) &= P(\text{one tail}) = P\{\text{th, ht}\} \\ &= \frac{2}{4} = \frac{1}{2} \geq 0 \end{aligned}$$

$$P(X=2) = P(\text{two tail}) = P\{\text{tt}\} = \frac{1}{4} \geq 0$$

$$P(X=0) + P(X=1) + P(X=2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

Thus the elementary probabilities associated with random variable

$X$  are  $\frac{1}{4}$ ,  $\frac{1}{2}$  and  $\frac{1}{4}$ .



(Q) A random variable  $X$  has the following probability distribution

$X$	0	1	2	3	4	5	6	7
$P(X)$	$a$	$4a$	$3a$	$7a$	$8a$	$10a$	$6a$	$5a$

- (i) Find the value of  $a$ .  
(ii) Find  $P(X < 3)$ ,  $P(X \geq 4)$  and  $P(0 < X < 5)$

Solution:-

$$(i) \sum_{i=1}^7 P_i = 1 \Rightarrow \sum P(X) = 1$$

$$\Rightarrow a + 4a + 3a + 7a + 8a + 10a + 6a + 5a = 1$$

$$\Rightarrow 44a = 1$$

$$\Rightarrow a = \frac{1}{44}$$

$$(ii) P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$= a + 4a + 3a = 8a = 8 \times \frac{1}{44} = \frac{2}{11}$$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + P(X=7)$$

$$= 8a + 10a + 6a + 5a = 29a$$

$$\Rightarrow 29 \times \frac{1}{44} = \frac{29}{44}$$

and  $P(0 < X < 5)$

$$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= 4a + 3a + 2a + 8a = 22a$$

$$= 22 \times \frac{1}{44} = \frac{22}{44} = \frac{1}{2}$$

Q) A fair dice is thrown once. Find the probability distribution of the random variable "getting an even number".

Solution: If a dice is thrown once, we may or may not get an even number. Let  $X$  be the random variable. Then  $X$  takes the value 1 if we get an even number and 0 if it's odd number.

$$\text{Thus } P(X=0) = P\{1, 3, 5\} = \frac{3}{6} = \frac{1}{2}$$

$$\text{and } P(X=1) = P\{2, 4, 6\} = \frac{3}{6} = \frac{1}{2}$$

We see that

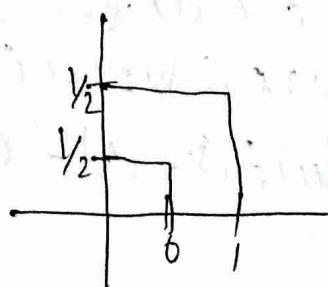
$$P(X=0) > 0 \text{ and } P(X=1) > 0 \text{ and}$$

$$P(X=0) + P(X=1) = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$

These are elementary probabilities associated with random variable  $X$ .

Therefore the probability distribution of  $X$ . i.e.  $(X, P(X))$ 's given by following

$X$	0	1
$P(X)$	$\frac{1}{2}$	$\frac{1}{2}$



## Expectation of a random variable or expected value or mathematical expectation

### Definition:-

If a discrete random variable  $X$  assumes the discrete set of values  $x_1, x_2, \dots, x_n$  with respective probabilities  $p_1, p_2, \dots, p_n$  where  $p_1 + p_2 + \dots + p_n = 1$ .

then the expectation or expected value or mathematical expectation of  $X$  is written as  $E(X)$  & is defined by

$$E(X) = x_1 p_1 + x_2 p_2 + x_3 p_3 + \dots + x_n p_n = \sum_{i=1}^n x_i p_i$$

$$\text{or } E(X) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \dots + p_n x_n = \sum_{i=1}^n p_i x_i$$

Similarly, the expected value of  $x^2$  is defined as

$$E(x^2) = \sum_{i=1}^n x_i^2 p_i$$

and the expected value of  $x^n$  is defined by

$$E(x^n) = \sum_{i=1}^n x_i^n p_i$$

### Properties:-

(I) If  $X$  is a random variable and  $a$  is constant

then (i)  $E(a) = a$

(ii)  $E(ax) = a E(X)$

(iii)  $E(X - a) = 0$

2) If  $x$  and  $y$  are two random variables then

$$E(x \pm y) = E(x) \pm E(y)$$

3)  $E(xy) = E(x) \cdot E(y)$  if  $x$  and  $y$  are two independent random variable.

4) if  $y = ax + b$  where  $a$  and  $b$  are constant  
then  $E(y) = aE(x) + b$ .

(Q) If  $X$  denotes the number of points in a dice.  
find the expectation of  $x$ .

Solution:-

Since  $x$  represents the number of points on a dice. The different values of  $x$  are  $1, 2, 3, 4, 5, 6$ . each having the same probability

$\frac{1}{6}$ .

By definition

$$\begin{aligned} E(X) &= \sum p_i x_i = P_1 x_1 + P_2 x_2 + P_3 x_3 + P_4 x_4 + P_5 x_5 + P_6 x_6 \\ &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 \\ &= \frac{1}{6} (1+2+3+4+5+6) \\ &= \frac{1}{6} \times \frac{6(6+1)}{2} = \frac{7}{2} = 3.5. \end{aligned}$$

(Q) find the probability distribution of  $X$ .  
the number of heads in the two tosses of a coin. and what is the expected value?

Solution:

In two tosses of a coin the number of head is 0, 1, 2.

$$\text{Now } P(X=0) = P(\text{no head}) = P\{\text{tt}\} = \frac{1}{4} > 0$$

$$P(X=1) = P(\text{one head}) = P\{\text{th, ht}\} = \frac{2}{4} = \frac{1}{2} > 0$$

$$P(X=2) = P(\text{two head}) = P\{\text{hh}\} = \frac{1}{4} > 0$$

$$P(X=0) > 0, P(X=1) > 0, P(X=2) > 0 \text{ then}$$

$$P(X=0) + P(X=1) + P(X=2)$$
$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

Thus these are elementary probabilities associated with random variable  $X$ .

Therefore the probability distribution of  $X$ . i.e.  $(X, P(X))$  is given by following table

$X$	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(X) = \sum P_i X_i = P_1 x_1 + P_2 x_2 + P_3 x_3$$

$$= \frac{1}{4}x_0 + \frac{1}{2}x_1 + \frac{1}{4}x_2$$

$$= 0 + \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

(Q)

$x$	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

find the expected value?

Solution:-

$$E(X) = \sum P_i x_i = \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4 + \dots + \frac{3}{36} \times 10 + \frac{2}{36} \times 11 + \frac{1}{36} \times 12 \\ = \frac{252}{36} = 7$$

Variance and Standard Deviation of a Random Variable:

Like mean, variance is a characteristic of a random variable  $X$  and it is used to measure dispersion or variation of  $X$ .

Variance :-

The variance of a random variable  $X$  is the expected value  $(X - \bar{x})^2$  where  $\bar{x}$  is the mean of  $X$ .

$\bar{x}$  is the mean of  $X$ .

i.e. variance of  $X$

$$\text{Var}(X) = E(X - \bar{x})^2$$

$$\text{then show that } \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\text{Def} \quad \text{Var}(x^2) = \sum p_i x_i^2 - \mu^2$$

## Standard Deviation (SD)

The standard deviation (S.D.) of a random variable  $x$  is the square root of the variance of  $x$ .

$$S.D.(x) = \sqrt{\text{Var}(x)} = \sqrt{E(x^2) - (E(x))^2}$$

(Q) A random variable  $x$  has the following probability distribution.

$x$	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the variance and hence determine the standard deviation of the distribution.

Solution:-

$$\text{Var}(x) = E(x^2) - \{E(x)\}^2$$

$$\text{Now } E(x) = \sum p_i x_i$$

$$= \frac{1}{8}x_0 + \frac{3}{8}x_1 + \frac{3}{8}x_2 + \frac{1}{8}x_3$$

$$= \frac{3+6+3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

$$\text{and } E(x^2) = \sum p_i x_i^2$$

$$= \frac{1}{8}x_0^2 + \frac{3}{8}x_1^2 + \frac{3}{8}x_2^2 + \frac{1}{8}x_3^2$$

$$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$$

$$\text{Var}(X) = 3 - (1.5)^2 = 3 - 2.25 = 0.75$$

Standard deviation ( $\sigma$ ) =  $\sqrt{\text{Var}(X)}$   
 $\sqrt{0.75} = 0.87$  (approximately)

(Q) For discrete probability distribution

$x$	0	1	2	3	4	5	6	7
$f(x)$	0	$K$	$2K$	$2K$	$3K$	$K^2$	$2K^2$	$7K^2 + K$

Determine (i)  $K$  (ii) mean (iii) variance  
 (iv) Standard Deviation.

Solution:

$$(i) \sum f(x) = 1$$

$$= 0 + K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$$\Rightarrow 10K^2 + 9K - 1 = 0$$

$$a = 10, b = 9, c = -1$$

$$K = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{(9)^2 - 4 \times 10(-1)}}{2 \times 10}$$

$$= \frac{-9 \pm \sqrt{81 + 40}}{20} = \frac{-9 \pm \sqrt{121}}{20}$$

$$K = \frac{-9+11}{20}$$

$$K = \frac{-9+11}{20} = \frac{2}{20} = \frac{1}{10}$$

$$K = \frac{-9-11}{20} = \frac{-20}{20} = -1$$

$$K = -1 \text{ or } \frac{1}{10}$$

$K = -1$  since probability can never be negative.

$$K = \frac{1}{10}$$

$x_i$	0	1	2	3	4	5	6	7
$f(x_i)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$$(i) \text{ Mean} = \sum x_i f(x_i) = E(X)$$

$$= 0 + 1 \times \frac{1}{10} + 2 \times \frac{2}{10} + 3 \times \frac{2}{10} + 4 \times \frac{3}{10} + 5 \times \frac{1}{100} \\ + 6 \times \frac{2}{100} + 7 \times \frac{17}{100}$$

$$= 3.66$$

$$E(X^2) = 0 + 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{2}{10} + 4^2 \times \frac{3}{10} + 5^2 \times \frac{1}{100} \\ + 6^2 \times \frac{2}{100} + 7^2 \times \frac{17}{100} \\ = \frac{1}{10} + \frac{4}{5} + \frac{9}{5} + \frac{24}{5} + \frac{1}{4} + \frac{92}{100} + \frac{49 \times 17}{100} \\ = 51.09$$

$$\begin{aligned}
 \text{(iii)} \quad \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
 &= \left[ 0 + 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{2}{10} + 4^2 \times \frac{3}{10} + 5^2 \times \frac{1}{100} + 6^2 \times \frac{2}{100} \right. \\
 &\quad \left. + 7^2 \times \frac{2}{100} \right] - (3.66)^2 \\
 &= 37.7
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \text{Standard deviation}(\sigma) &= \sqrt{\text{Var}(X)} \\
 &= \sqrt{37.7} = 6.14
 \end{aligned}$$

(Q) A random variable has the following probability distribution

$X$	4	5	6	8
$P(X)$	0.1	0.3	0.4	0.2

Find the expectation and the standard deviation of the random variable.

Solution:-

$$\text{Expectation of } X = E(X) = \sum P_i \cdot x_i$$

$$\begin{aligned}
 &= 0.1 \times 4 + 0.3 \times 5 + 0.4 \times 6 + 0.2 \times 8 \\
 &= 0.4 + 1.5 + 2.4 + 1.6 = 5.9
 \end{aligned}$$

$$E(X^2) = \sum P_i \cdot x_i^2$$

$$\begin{aligned}
 &= 0.1 \times (4)^2 + 0.3 \times (5)^2 + 0.4 \times (6)^2 + 0.2 \times (8)^2 \\
 &= 0.1 \times 16 + 0.3 \times 25 + 0.4 \times 36 + 0.2 \times 64 \\
 &= 1.6 + 7.5 + 14.4 + 12.8 = 36.3
 \end{aligned}$$

$$\begin{aligned}
 \text{var}(X) &= E(X^2) - \{E(X)\}^2 \\
 &= 36.3 - (5.9)^2 \\
 &= 36.3 - 34.81 = 1.49 \\
 \text{Hence standard deviation}(\sigma) &= \sqrt{\text{var}(X)} \\
 &= \sqrt{1.49} = 1.22
 \end{aligned}$$

Moment Generating Function (M.G.F.) :-

The moment generating function (m.g.f.) of a random variable  $X$  about the origin is defined as

$$M_0(t) = E(e^{tX}) = \left\{ e^{tx} P(x) \right\}$$

This function  $M_0(t)$  is called moment generating function because all the moments of  $X$  can be obtained from  $M_0(t)$  as follows

$$\begin{aligned}
 M_0(t) &= \left\{ 1 + tx + \frac{t^2}{2!} x^2 + \frac{t^3}{3!} x^3 + \dots + \frac{t^n}{n!} x^n \right\} P(x) \\
 &= 1 + t E(x) + \frac{t^2}{2!} E(x^2) + \dots + \frac{t^n}{n!} E(x^n) t^n \\
 &= 1 + t \mu_1 + \frac{t^2}{2!} \mu_2 + \dots + \frac{t^n}{n!} \mu_n
 \end{aligned}$$

where  $\mu_n = E(x^n)$

→ If  $x$  and  $y$  are two independent random variables the m.g.f. of  $x+y$  is given by

$$M_{x+y}(t) = M_x(t) + M_y(t)$$