

IV: 4

Factorial:

The continued product of first n natural numbers is called the " n factorial" and it is denoted by $n!$ or b .

$$\text{e.g. } n! = 1 \times 2 \times 3 \times 4 \times \dots \times (n-1) \times n$$

$$\text{Ex: } 3! = 1 \times 2 \times 3 = 6$$

$$4! = 1 \times 2 \times 3 \times 4 = 24$$

$$\text{Then } n! = n \times (n-1)!$$

$$4! = 4 \times 3! = 4 \times 3! = 4 \times 3 \times 2 \times 1 = 24$$

$$0! = 1 \text{ (zero factorial)}$$

Note: Factorials of proper fractions or negative integers are not defined. Factorial n is defined only for whole numbers.

Permutations:

→ A permutations of given things (elements or objects) is an arrangement of these things in a row in some order.

→ The number of different permutation of n different things taken K at a time with out repetition is

$$n(n-1)(n-2) \dots (n-K+1) = \frac{n!}{(n-K)!}$$

$$\text{or } P(n, k) = {}^n P_k = \frac{n!}{(n-k)!}$$

and with repetitions is n^K .

Combination:-

A combination of given things means any selection of one or more things with or without regard to order. There are two kinds of combinations; as follows

→ The number of combinations of n different things taken r at a time with or without repetitions is

$$C(n, r) \text{ or } {}^n C_r \text{ or } {}^n C_{(r)} = \frac{n!}{r!(n-r)!}$$

and with repetitions is $\binom{n+r-1}{r}$

Binomial Distribution or Bernoulli Distribution

→ Binomial distribution is a discrete probability distribution which is obtained from a success or failure. It comes in an experiment or survey that is repeated multiple times.

→ The binomial is a type of distribution that has two possible outcomes (by means two or two)

→ For a binomial distribution the probability function is $p(X=r) = {}^n C_r p^r q^{n-r}$

where p or q are parameters (success or failure)
 n = the number of time the experiment repeats.

The parameter n is always a true integer.

→ Binomial distribution of mean = np ,
 variance = npq
 and standard deviation = \sqrt{npq}

(Q) Find the probability of getting 4 heads in 6 tosses of a fair coin.

Solution:-

$$P = \frac{1}{2}, q = \frac{1}{2}, n = 6, r = 4$$

$$\begin{aligned} P(X=r) &= {}^n C_r p^r q^{n-r} \\ &= {}^6 C_4 \cdot p^4 q^{6-4} \\ &= \frac{6!}{4!(6-4)!} \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^2 \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1} \times \frac{1}{16} \times \frac{1}{4} \end{aligned}$$

$$= \frac{15}{64}$$

(Q) Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution:-

When one coin is thrown

The probability of getting a head = $\frac{1}{2}$

$$P = \frac{1}{2}$$

The probability of not getting a head = $1 - \frac{1}{2} = \frac{1}{2}$

$$q = \frac{1}{2}$$

$$\begin{aligned} \text{Then } P(\text{at least 7 heads}) &= P(6 \text{ heads}) + P(8 \text{ heads}) + P(9 \text{ heads}) \\ &\quad + P(10 \text{ heads}) \end{aligned}$$

$$\begin{aligned}
 &= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) \\
 &\quad + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \\
 &= \frac{1}{2^{10}} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right] \\
 &= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024} = \frac{11}{64}
 \end{aligned}$$

H.W. 8 coins are tossed simultaneously.
 Q) Find the probability of getting at least 6 heads. [Ans: $\frac{37}{256}$]

Q) If the sum of the mean and the variance of binomial distribution of 5 trials is 4.8, find the distribution.

Solution:-
 Let the required binomial distribution be ${}^n P^n q^{n-n}$.

Where n = number of trials = 5

Mean of the distribution = NP

and variance of the distribution = NPq

By the given condition

$$NP + NPq = 4.8$$

$$5P + 5Pq = 4.8$$

$$\Rightarrow 5P(1+q) = 4.8$$

$$\Rightarrow 5(1-q)(1+q) = 4.8$$

$$(1-p+q=1) \\ p=1-q$$

$$\Rightarrow 5 \times 10 (1-q^2) = 48 \times 10$$

$$\Rightarrow 50(1-q^2) = 48$$

$$\Rightarrow 50 - 50q^2 = 48$$

$$\Rightarrow -50q^2 = 48 - 50 = -2$$

$$\Rightarrow 50q^2 = 2$$

$$\Rightarrow q^2 = \frac{2}{50} = \frac{1}{25}$$

$$\Rightarrow q = \frac{1}{5}$$

$$\Rightarrow p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$$

Hence the required binomial distribution

$$5 \cdot \binom{4}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{5-k}$$

Poisson Distribution:-

Poisson distribution is a particular limiting form of the binomial distribution when p or q is very small and n is large enough.

Poisson distribution is

$$P(X) = \frac{m^x \cdot e^{-m}}{x!}$$

$$\text{or } P(X) = \frac{e^{-m} \cdot m^x}{x!}$$

This is the probability of x success for poisson distribution. For $x=0, 1, 2, 3, \dots$ we get the probabilities of $0, 1, 2, 3, \dots$ successes as

$$P(0) = e^{-m}, P(1) = m e^{-m}$$

$$P(2) = \frac{m^2}{2!} e^{-m}, P(3) = \frac{m^3}{3!} e^{-m} \dots \text{and so on.}$$

Note: (1) The sum of probabilities $P(n)$ is 1
for $n = 0, 1, 2, 3, \dots$

$$\begin{aligned} \text{e.g. } P(n) &= P(0) + P(1) + P(2) + P(3) + \dots \\ &= e^{-m} \left(1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right) \\ &= e^{-m} \times e^m = e^{-m+m} = e^0 = 1 \end{aligned}$$

(2) Poisson distribution possesses only one parameter m . If we consider the length of interval as d instead of unit length. The average number of occurrences in d length of interval is md thus the probability function of Poisson distribution is

$$P(n) = \frac{e^{-md} (md)^n}{n!}$$

(Q) The number of telephone calls arriving on an internal switch board of an office is 90 per hour. Find the probability that at the most 1 to 3 calls in a minute on the board arrive (use $e^{-1.5} = 0.223$)

Solution:

$$\text{Here } m = \text{mean} = \frac{90}{60} = 1.5$$

X will follow Poisson distribution

Now the probability that at the most 1 to 3 calls in one minute $= P(0) + P(1) + P(2) + P(3)$

Where $P(n) = \frac{e^{-m} \cdot m^n}{n!}$

$$P(1) = \frac{e^{-1.5}(1.5)^1}{1!} = 1.5 \times e^{-1.5} = 1.5 \times 0.223$$

$$P(2) = \frac{e^{-1.5}(1.5)^2}{2!} = 1.125 \times e^{-1.5} = 1.125 \times 0.223$$

$$P(3) = \frac{e^{-1.5} \cdot (1.5)^3}{3!} = 0.5625 \times e^{-1.5} = 0.5625 \times 0.223$$

$$\text{Required probability} = 0.223(1.5 + 1.125 + 0.5625) \\ = 0.710$$

(Q) Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book. What is the probability that 10 pages, selected random, will be free from errors?

Sol: Here $p = \frac{43}{585} = 0.0735$ and $n = 10$.

$$\therefore n = np = 10 \times 0.0735 = 0.735$$

Clearly, p is very small and n is large.

So it is a case of Poisson distribution.

Let X denote the number of errors in 10 pages.

$$\text{Then } P(X=n) = \frac{e^{-0.735} \cdot (0.735)^n}{n!}$$

$$P(X = \text{no error}) = P(X = 0) = \frac{e^{-0.735} \times (0.735)^0}{0!}$$

$$= e^{-0.735} = 0.4795.$$

Hence the required probability is 0.4795

Hypergeometric Distribution:-

Suppose we have a lot of N items out of which r items are defective and the rest $(N-r)$ are non-defective. Let us choose at random n items from the lot with or without replacement. Let X be the number of defective found.

$$\text{Then } P(X=k) = \frac{\binom{r}{k} \binom{N-r}{n-k}}{\binom{N}{n}}, k=0, 1, 2, \dots$$

A discrete random variable having probability distribution is said to have a hypergeometric distribution.

Uniform Distribution:-

- Uniform distribution are probability distributions with equally likely outcomes
- There are two types of uniform distributions such as

(i) Discrete uniform distributions.

(ii) Continuous uniform distributions.

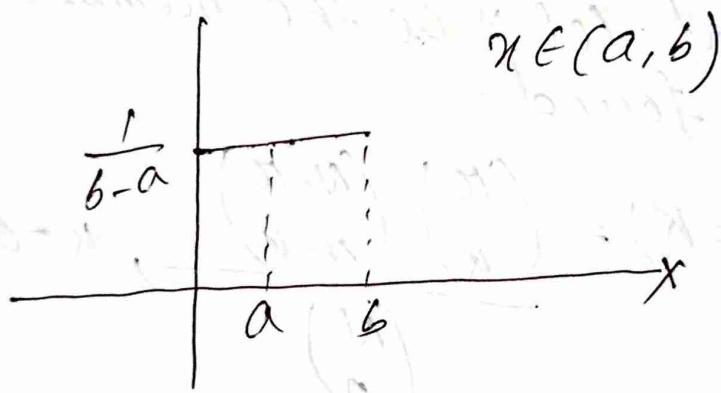
Ex: A coin also has a uniform distribution because the probability of getting either heads or tails in a coin toss is same.

→ The distribution is the easiest of all the continuous distribution. It is given by

$$f(x) = \frac{1}{b-a}; \quad a < x < b$$

$$= 0, \quad \text{otherwise}$$

Graph



(i) Discrete uniform distributions:-

Discrete uniform distribution is a symmetric probability distribution where in a finite number of values are equally likely to be observed; every one of n values has equal probability $\frac{1}{n}$.

→ Notation: $\{a, b\}$ are consecutive integers with $b > a$

Parameters: a, b integers with $b > a$
 $n = b - a + 1$

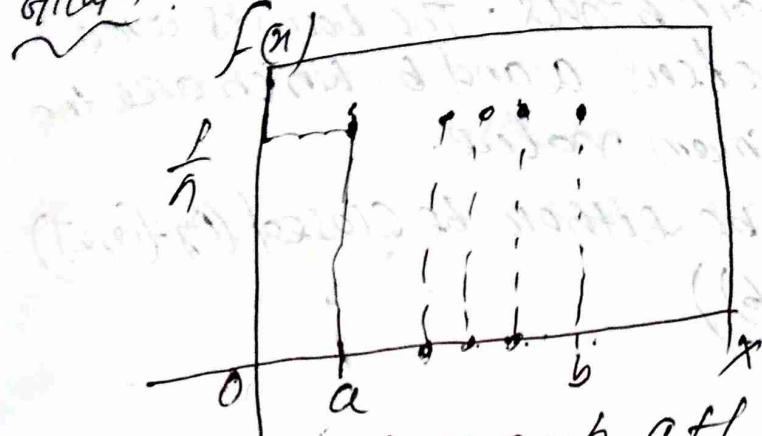
probability mass function (PMF) = $\frac{1}{n}$

mean = $\frac{a+b}{2}$

Median = $\frac{a+b}{2}$

Variance = $\frac{(b-a+1)^2 - 1}{12}$

Graph:



$n=5$, where $n=b-a+1$

$n=5-1+1=5$

- Q) If $a=4$, $b=6$ find (i) PMF (ii) mean
(iii) median (iv) variance

Sol:

Given that $a=4$, $b=6$

$n=b-a+1=6-4+1=3$

(i) PMF = $\frac{1}{n} = \frac{1}{3} = 0.3$

(ii) Mean = $\frac{a+b}{2} = \frac{4+6}{2} = \frac{10}{2} = 5$

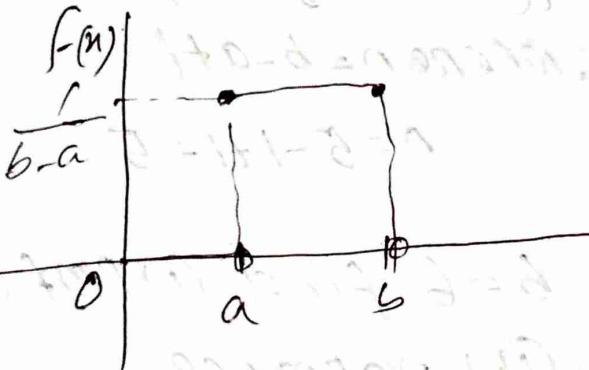
(iii) Median = $\frac{a+b}{2} = \frac{4+6}{2} = 5$

(iv) Variance = $\frac{(b-a+1)^2 - 1}{12} = \frac{(3^2 - 1)}{12} = \frac{9-1}{12} = \frac{8}{12}$

(ii) Continuous Uniform Distribution:

- Continuous uniform distribution or rectangular distribution is a family of symmetric probability distributions
- The distribution describes an experiment where there is an arbitrary outcome that lies between certain bounds. The bounds are defined by parameters a and b , which are the minimum and maximum value.
- The intervals can be either be closed (eg.: $[a, b]$) or open (eg.: (a, b))

Graph:



Notation:- $\sim U(a, b)$ or uniform (a, b)

parameters: $a \leq b < \infty$

PDF:
$$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Mean: $\frac{a+b}{2}$

Median: $\frac{a+b}{2}$

Variance: $\frac{(b-a)^2}{12}$

Normal Distribution: (or Gaussian or Gauss or Laplace-Gauss)
A normal distribution is a type of continuous probability distribution for a real-valued random variable.

Notation: $N(\mu, \sigma^2)$

parameters: $\mu \in \mathbb{R}$ = mean

$\sigma^2 > 0$ = variance

$$\text{PDF} : f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, \sigma > 0$$

Mean: μ

Median: μ

Variance: σ^2

Standard normal variate: $Z = \frac{x-\mu}{\sigma}$

Q) If $\mu = 50$ and $\sigma = 10$ find

(i) $P(50 \leq X \leq 80)$ (ii) $P(60 \leq X \leq 70)$. Use

Table: Area under the normal curve

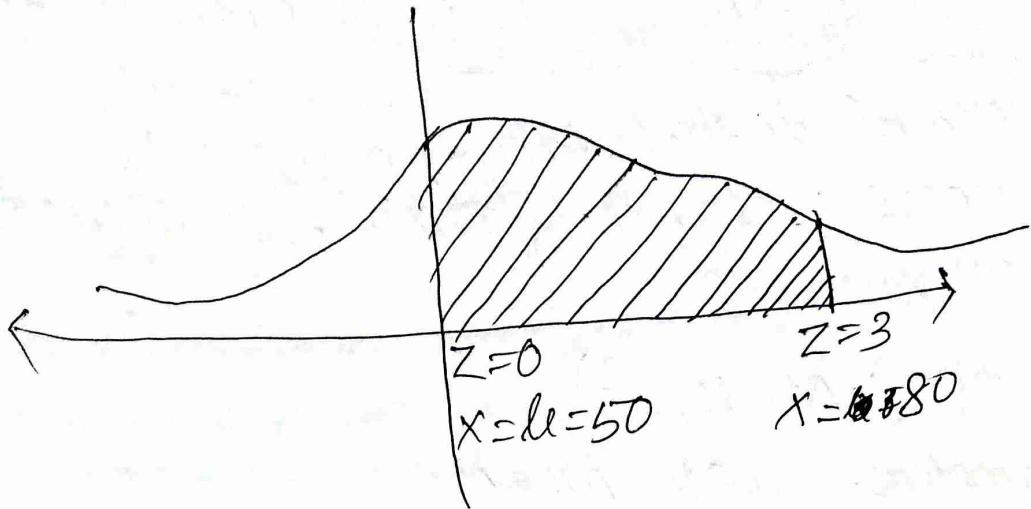
Solution:

standard normal variate $Z = \frac{x-\mu}{\sigma} = \frac{x-50}{10}$

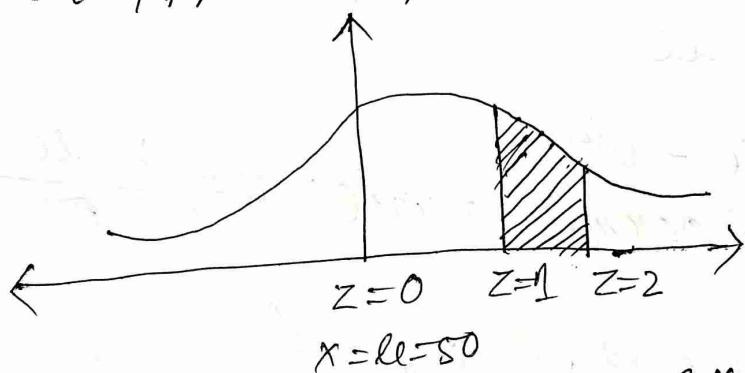
(i) $Z = \frac{50-50}{10} = 0$. When $X=50$.

and $Z = \frac{80-50}{10} = 3$ when $X=80$

Hence $P(50 \leq X \leq 80) = P(0 \leq Z \leq 3) = 0.4987$



$$\begin{aligned}
 \text{(ii)} \quad P(60 \leq X \leq 70) &= P(1 \leq Z \leq 2) = \text{Area from } Z=1 \text{ to } Z=2 \\
 &= (\text{Area from } Z=0 \text{ to } Z=2) - (\text{Area from } Z=0 \text{ to } Z=1) \\
 &= 0.4772 - 0.3413 = 0.1359
 \end{aligned}$$



(Q) Suppose 10 percent of probability rate α normal distribution $N(\mu, \sigma^2)$ is below 25 and 5 percent above 90, what are the value of μ and σ .

$$\text{Sol: } P(X \leq 35) = \frac{10}{100} = 0.1 \quad \text{--- (i)}$$

$$P(X > 90) = \frac{5}{100} = 0.05 \quad \text{--- (ii)}$$

The standard normal variable $Z = \frac{X - \mu}{\sigma}$

$$\text{When } x = 35 \text{ then } Z = \frac{35 - \mu}{\sigma}$$

When $x=90$ then $Z = \frac{90-\mu}{\sigma}$

from eqn(i) $P(x \leq 35) = 0.1$

$$\Rightarrow P\left(Z \leq \frac{35-\mu}{\sigma}\right) = 0.1$$

$$\frac{(35-\mu)}{\sigma} = -1.28 \text{ from table}$$

$$\Rightarrow \mu - 1.28\sigma = 35 \quad (\text{iii})$$

From eqn(ii) $P(x > 90) = 0$

$$P\left(Z > \frac{90-\mu}{\sigma}\right) = 0.05$$

$$\Rightarrow \frac{90-\mu}{\sigma} = 1.645 \text{ from table}$$

$$\Rightarrow 90 - \mu = 1.645\sigma$$

$$\Rightarrow \mu - 1.645\sigma = 90 \quad (\text{iv})$$

From eqn(iii) and (iv) we get

$$\mu = 157.87$$

$$\sigma = 150.68$$

Hence $\mu = 157.87$, $\sigma = 150.68$ (Ans)

Bivariate or joint probability Distribution

Joint probability:-

Two random variable X and Y are said to be jointly distributed if they are defined on same probability space. The Joint probability function is denoted by $P_{XY}(x, y)$ or $f_{XY}(x, y)$.

Joint probability Mass function:-

Let X and Y be random variables on a sample space S with respective image sets

$$X(S) = \{x_1, x_2, \dots, x_n\} \text{ and } Y(S) = \{y_1, y_2, \dots, y_m\}$$

The function p on $X(S) \times Y(S)$ defined as

$p_{ij} = P(X = x_i \cap Y = y_j) = P(x_i, y_j)$ is called joint probability function of X and Y

$$\text{where } X(S) \times Y(S) = \{x_1, x_2, \dots, x_n\} \times \{y_1, y_2, \dots, y_m\}$$

Table:-

		y_1	y_2	y_3	\dots	y_m	Total
		$x \rightarrow$	x_1	x_2	x_3	\dots	x_n
$y \downarrow$	x_1	p_{11}	p_{12}	p_{13}	\dots	p_{1m}	p_1
	x_2	p_{21}	p_{22}	p_{23}	\dots	p_{2m}	p_2
$x \rightarrow$	x_3	p_{31}	p_{32}	p_{33}	\dots	p_{3m}	p_3
	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
		\vdots	\vdots	\vdots	\ddots	p_{mn}	p_n
		x_1	p_{n1}	p_{n2}	p_{n3}	\dots	p_m
Total		P_1	P_2	P_3	\dots	P_m	

Marginal and conditional probability function -

Suppose the joint distribution of two random variable X and Y is given then the probability distribution of X is determined as follows

$$f_X(x_i) = P_X(x_i) = P(X=x_i) = p_{i1} + p_{i2} + \dots + p_{im}$$
$$= \sum_{j=1}^m p_{ij} = p_i$$

and is known as marginal probability function of X .

Similarly $f_Y(y_j) = P_Y(y_j) = \sum_{i=1}^n p_{ij} = p_j$
is called marginal probability function of Y .
Also $f_{X|Y}(x_i|y_j) = P(X=x_i|Y=y_j) = \frac{P(X=x_i \cap Y=y_j)}{P(Y=y_j)}$

$$= \frac{P(x_i, y_j)}{P(y_j)} = \frac{p_{ij}}{p_j}$$

This called conditional probability function of X when $Y=y_j$ is given

similarly

$$f_{Y|X}(y_j|x_i) = \frac{P(Y=y_j | X=x_i)}{P(X=x_i)}$$
$$= \frac{P(x_i, y_j)}{P(x_i)} = \frac{p_{ij}}{p_i}$$

is conditional probability function of Y when $X=x_i$ is given.

- (Q) For the following bivariate probability distribution of X and Y find:
- $P(X \leq 2, Y=3)$
 - $P(Y=4)$
 - $P(X \leq 1)$
 - $P(Y \leq 5)$

$X \backslash Y$	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution:-

$X \backslash Y$	1	2	3	4	5	6	$P_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$P_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1

$$\begin{aligned}
 (i) P(X \leq 2, Y=3) &= P(X=0, Y=3) + P(X=1, Y=3) + \\
 &= \frac{1}{32} + \frac{1}{8} + \frac{1}{64} = \frac{2+8+1}{64} = \frac{11}{64}
 \end{aligned}$$

$$(ii) P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$$

$$(iii) P(Y=Y) = \frac{13}{64}$$

$$(iv) P(Y \leq 5) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)$$

$$+ P(Y=4) + P(Y=5)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} + \frac{13}{64} + \frac{6}{32}$$

$$= \frac{6+6+11+13+12}{64} = \frac{48}{64}$$

Exponential Distribution:

A random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0; & \text{otherwise} \end{cases}$$

Remark: Sometimes exponential distribution is defined by the p.d.f

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x > 0 \\ 0; & \text{otherwise} \end{cases}$$

(e) Graph of Exponential probability density function

$$\text{for } f(x) = \lambda e^{-\lambda x}$$

x	0	1	2	\dots	∞
$f(x)$	λ	$\lambda e^{-\lambda}$	$\lambda e^{-2\lambda}$	\dots	0

when $\lambda = 1$, $f(x) = 1, e^{-1}, e^{-2}, \dots 0$

$\lambda = 2$, $f(x) = 2, 2e^{-2}, 2e^{-4}, \dots 0$

